Sensor Localization from WLS Optimization with Closed-form Gradient and Hessian

Giuseppe Destino and Giuseppe Thadeu Freitas de Abreu
Center for Wireless Communications
University of Oulu, Finland
E-mails: [giuseppe.destino,giuseppe.abreu]@ee.oulu.fi

***Abstract*— A non-parametric, low-complexity algorithm for accurate and simultaneous localization of multiple sensors from scarce and imperfect ranging information is proposed. The technique is based on multidimensional scaling (MDS) and weighted least-squares (WLS) optimization. Closed-form expressions of the gradient and Hessian of the weighted quadratic objective used to solve the WLS problem are also provided. The performance of the proposed technique is studied through extensive computer simulations, with the intra-node distances randomly generated in accordance to a statistical model constructed from the results of a measurement campaign conducted with a pair of impulsive ultra-wideband (UWB) radios in an indoor scenario. The simulation results reveal that the proposed algorithm, despite its low complexity, is nearly as accurate as the known alternative of best performance, which is based on semi-definite programming and demands significantly more computational power.**

I. Introduction

The problem of estimating the location of multiple points from the corresponding set of mutually-exclusive pairwise distances has been long ago considered in areas as distinct as psychology, geography and molecular biology, and typically solved by variations of the so-called multidimensional scaling (MDS) technique [1]. When the metric used to measure the distance between points is the Euclidean distance, the matrices containing all pairwise distances is known as the *Euclidean distance matrix* (EDM) and the corresponding MDS technique is referred to as the *metric* MDS algorithm.

The idea of employing MDS to estimate the location of wireless transceivers in ad-hoc and sensor networks, however, has only recently been proposed [2]–[5].

In comparison to "classic" applications of MDS [1], it can be said that this new form of the problem brings fresh and perhaps harsher conditions. Indeed, problems such as molecular conformation [6] admit computationally demanding solutions meant to be calculated sporadically, with the assistance of powerful computers. Furthermore, the metric between pairs of points are, in most classical applications, measured by the same technique and, therefore, follow similar statistics. Finally, no more than a couple of EDM samples are typically considered in classical applications [1], [6].

In contrast, typical wireless network localization applications require the simultaneous and frequently-updated computation of multiple source locations, carried out by platforms of limited computational power, from imperfect and incomplete ranging information collected by non-identical devices.

This is the problem dealt with in this paper. In particular, a fast, robust and accurate algorithm, that exploits the entire set of incomplete and imperfect EDM samples is proposed.

The algorithm is based on the combination of MDS and weighted least-squares (WLS) minimization, used to complete and approximate imperfect and incomplete EDM samples. The approach, being entirely non-parametric, does not put any constraints to ranging method, therefore, it enables its application to heterogeneous networks.

The remainder of the paper is structured as follows. In section II, the problem and assumptions are stated formally and put into context. Terminology and mathematical notations, used throughout the paper, are defined in the same section. In section IV the proposed algorithm is described in detail. The performance of the technique is studied in section V and a brief conclusion is given in section VI.

II. Problem Statement

A. Localization and Multidimensional Scaling

We consider a network of N devices embedded in the ε-dimensional Euclidean space. Such a network can be represented [7] by a simplicial graph $G_{\varepsilon,N}(\mathbf{p},\mathbf{E})$, with vertices $\mathbf{p}$, edges $(n,m) \in \mathbf{E}$ and weights $\{d_{n,m} : (n,m) \in \mathbf{E}\}$. Edges represent communication links, while weights correspond to the Euclidean distance between sensors, given by

$$d_{n,m} = \mathcal{D}([\mathbf{x}_n, \mathbf{x}_m]) \triangleq \sqrt{\langle(\mathbf{x}_n - \mathbf{x}_m); (\mathbf{x}_n - \mathbf{x}_m)\rangle}, \quad (1)$$

where $\mathbf{x}_n$ denotes the coordinate vector of p_n and $\langle \mathbf{x}_n; \mathbf{x}_m \rangle$ the inner product of $\mathbf{x}_n$ and $\mathbf{x}_m$, with $\mathbf{x}_n \neq \mathbf{x}_m \ \forall \ n \neq m$.

The graph has the associated EDM $\mathbf{D} \triangleq [d_{n,m}] \in \mathbb{S}^{N\times N}$, where $\mathbb{S}$ denotes the set of hollow symmetric matrices.

Let $\mathbf{C}$ be the binary-valued adjacent matrix of the graph $G_{\varepsilon,N}(\mathbf{p},\mathbf{E})$, with $c_{n,m} = 1$ and $c_{n,m} = 0$ indicating the existence and the non-existence of a link between p_n and p_m.

The matrix $\mathbf{C}$ will be referred as *connectivity matrix* of the network. If we consider a *complete* graph, the associated connectivity matrix is equal to $\mathbf{C}_\text{f} = \mathbf{1} \cdot \mathbf{1}^\text{T} - \mathbf{I}$, where the superscript $^\text{T}$ denotes transpose, $\mathbf{1}$ is the all-one column vector and $\mathbf{I}$ is the identity matrix. Such a network (graph) is said to be *fully connected*.

The term *completeness* is used in reference to the ratio ϱ of the cardinality of the set $\mathbf{E}$ and the set $\mathbf{E}_\text{f}$ of the corresponding complete graph. Likewise, the term *incompleteness* will be used in reference to the complement of such a ratio.

Mathematically we have

$$\varrho = |\mathbf{E}|/|\mathbf{E}_{\mathrm{f}}|, \tag{2}$$

where $|\cdot|$ denotes cardinality.

A matrix $\mathbf{S} \in \mathbb{S}^{N\times N}$ is said to be a *true* EDM in the embedding ε if and only if (iff),

$$\exists\, \mathbf{X} = [\mathbf{x}_1, \cdots, \mathbf{x}_N] \in \mathbb{R}^{\varepsilon\times N} \mid \mathbf{S} = \mathcal{D}(\mathbf{X}), \tag{3}$$

where the slightly abused notation $\mathbf{S} = \mathcal{D}(\mathbf{X})$ indicates that $s_{n,m} = \mathcal{D}([\mathbf{x}_n, \mathbf{x}_m])\ \forall\ (n,m)$.

The set of all true N-dimensional EDMs in the ε-dimensional Euclidean space will be denoted by $\mathbb{E}^{N\times N}$. A matrix $\tilde{\mathbf{D}} \in \mathbb{S}^{N\times N}, \tilde{\mathbf{D}} \notin \mathbb{E}^{N\times N}$ that is close to a true EDM (in the Frobenius norm sense) will be referred as an EDM *sample*.

Let $\{\tilde{\mathbf{D}}\} = \{\tilde{\mathbf{D}}_1, \cdots, \tilde{\mathbf{D}}_K\}$ denote the set of EDM samples $\tilde{\mathbf{D}}_k$. Then, the positioning problem under consideration can be mathematically stated as follows:

$$\min_{\hat{\mathbf{X}}\in\mathbb{R}^{\varepsilon\times N}} f(\hat{\mathbf{X}}) = ||\mathcal{G}(\{\tilde{\mathbf{D}}\}) - \mathcal{G}(\mathcal{D}(\hat{\mathbf{X}}))||_{\mathrm{F}}^2, \tag{P – 1}$$

where $||\cdot||_{\mathrm{F}}^2$ denotes the squared Frobenius norm and $\mathcal{G}$ is a function to be determined.

III. Background

The first difficulty in posing and solving problem (P – 1) is to choose an appropriate function for $\mathcal{G}$. In the special case where the true EDM – rather then its samples $\tilde{\mathbf{D}}_k$ – is known, problem (P – 1) becomes

$$\min_{\hat{\mathbf{X}}\in\mathbb{R}^{\varepsilon\times N}} f(\hat{\mathbf{X}}) = ||\mathbf{D} - \mathcal{D}(\hat{\mathbf{X}})||_{\mathrm{F}}^2, \tag{P – 2}$$

Problem (P – 2) is equivalent to that solved exactly, $i.e.$, with $f(\hat{\mathbf{X}}) = 0$, by the metric MDS algorithm [1], which can be summarized as:

$$\hat{\mathbf{X}} = \left([\mathbf{U}]_{\mathrm{UL}:N\times\varepsilon} \cdot [(\mathbf{\Lambda})^{\frac{1}{2}}]_{\mathrm{UL}:\varepsilon\times\varepsilon}\right)^{\mathrm{T}}, \tag{4}$$

where $[\cdot]_{\mathrm{UL}:n\times m}$ denotes the n-by-m upper-left partition, $(\cdot)^m$ denotes element-wise (Hadamard) power of m-th order and the matrices $\mathbf{U}$ and $\mathbf{\Lambda}$ in equation (4) are the eigenvector and eigenvalue matrices (both in decreasing order) of the positive semi-definite Euclidean kernel defined as [8]

$$\mathbf{K} \triangleq \mathcal{K}(\mathbf{D}) = -(\mathbf{J}\cdot(\mathbf{D})^2\cdot\mathbf{J})/2, \tag{5a}$$
$$\mathbf{J} \triangleq \mathbf{I} - (\mathbf{1}\cdot\mathbf{1}^{\mathrm{T}})/N. \tag{5b}$$

A. Completion and Approximation of EDM samples

Problem (P – 2) is a trivial variation of (P – 1), and its application to source localization in wireless networks is limited, since disturbances affecting the EDM sample – in particular erasure of some entries of $\{\tilde{\mathbf{D}}\}$ (incompleteness) – have a devastating effect on positioning algorithms based on a straightforward application of MDS.

Non-surprisingly, a considerable amount of work has been poured into the approximation and completion of imperfect EDM samples [9], hereafter referred to as EDM-AP and EDM-CP, respectively. Traditionally, this problem is solved by non-linear optimization over an objective function designed to exploit certain properties of EDMs [8].

In [10], for instance, Wolkowicz *et al.* proposed to pose the EDM-AP and EDM-CP in the form:

$$\min_{\substack{\hat{\mathbf{W}}\in\mathbb{R}^{(N-1)\times(N-1)}\\ \hat{\mathbf{W}}\succeq 0}} ||\mathbf{H}\circ(\tilde{\mathbf{D}})^2 - \mathbf{H}\circ\mathcal{W}^{-1}(\hat{\mathbf{W}})||_{\mathrm{F}}^2, \tag{P – 3}$$

where the notation $\mathbf{M}\succeq 0$ indicates positive semi-definiteness of $\mathbf{M}$, $\circ$ denotes *Hadamard* product and $\mathbf{W}$ is defined as,

$$\mathbf{W} = \mathcal{W}(\mathbf{D}) = -(\mathbf{V}^{\mathrm{T}}\cdot(\mathbf{D})^2\cdot\mathbf{V})/2, \tag{6a}$$

$$\mathbf{V} = \begin{bmatrix} -\frac{1}{\sqrt{N}}\mathbf{1} \\ \cdots\cdots\cdots\cdots\cdots \\ \mathbf{I} - \frac{1}{N+\sqrt{N}}\mathbf{1}\cdot\mathbf{1}^{\mathrm{T}} \end{bmatrix}. \tag{6b}$$

It was shown in [10] that $\mathbf{W}$ is a positive semi-definite kernel of the Euclidean distance matrix $\mathbf{D}$, that admits the unique inverse mapping $\mathbf{D} = (\mathcal{W}^{-1}(\mathbf{W}))^{\frac{1}{2}}$, where

$$\mathcal{W}^{-1}(\mathbf{W}) \triangleq \mathrm{diag}(\mathbf{W}_{\mathrm{V}})\cdot\mathbf{1}^{\mathrm{T}} + \mathbf{1}^{\mathrm{T}}\cdot\mathrm{diag}(\mathbf{W}_{\mathrm{V}}) - 2\mathbf{W}_{\mathrm{V}} \tag{7}$$

with $\mathbf{W}_{\mathrm{V}} = \mathbf{V}\cdot\mathbf{W}\cdot\mathbf{V}^{\mathrm{T}}$.

The weight matrix $\mathbf{H}$ in problem (P – 3) is inserted in order to control the variation of each element of the solution. To the best of our knowledge, however, no optimality theory for the weights has been derived. It is only conjectured in [10] that the entries of $\mathbf{H}$ should hold a proportion to the confidence on the corresponding entries of the given EDM sample $\tilde{\mathbf{D}}$, with $h_{n,m} = 0$ for $\tilde{d}_{n,m}$ unknown, and $h_{n,m} \gg 0$ indicating that $\tilde{d}_{n,m}$ is accurately measured.

The idea behind this formulation is to exploit the positive semi-definiteness of the kernel $\mathbf{W}$ as an intrinsic constraint onto the solution. The problem can then be solved by optimization over the positive semi-definite cone, with certain convergence advantages [10]. The method is powerful and solves both the EDM-AP and the EDM-CP simultaneously. Unfortunately, the method also requires extensive computational power, which limits its application in our context.

The following low-complexity Least-Squares (LS) formulation of the EDM-AP has been proposed by Chu *et al.* [11]:

$$\min_{\hat{\mathbf{X}}\in\mathbb{R}^{\varepsilon\times N}} ||(\tilde{\mathbf{D}})^2 - (\mathcal{D}(\hat{\mathbf{X}}))^2||_{\mathrm{F}}^2. \tag{P – 4}$$

This formulation enables Newton optimization methods to be employed in the solution of EDM-AP. The authors provide closed-form expressions for the gradient and Hessian of the cost function in (P – 4), which further contributes to reducing the complexity of the method. Although the objective of problem (P – 4) is convex on space of solutions $\hat{\mathbf{X}}$, it is empirically shown that the method is convergent. The substantial gain in complexity reduction, however, makes up for this mathematical drawback.

Unfortunately, Chu's approach suffers from the same sensitivity to incompleteness of the MDS algorithm.

IV. EDM Completion and Approximation from Weighted Least Squares Optimization

The following can be learnt about the EDM-AP/CP by studying the similarities and differences of the two approaches mentioned above.

On the one hand, the matrix $\mathbf{H}$ plays a central role in the efficacy of Wolkowicz' method [10], [12], allowing each element of the EDM sample to be weighted according to its relevance. On the other hand, the positive semi-definiteness constraint is both cumbersome and less important as N increases.

Therefore, in order to combine the advantages of both methods, we propose to state the positioning problem as

$$\min_{\hat{\mathbf{X}} \in \mathbb{R}^{\varepsilon \times N}} \left|\left| \mathcal{H}(\{\tilde{\mathbf{D}}\}) \circ \left((\bar{\mathcal{G}}(\{\tilde{\mathbf{D}}\})^2 - (\hat{\mathbf{D}})^2 \right) \right|\right|_{\text{F}}^2, \tag{P – 5}$$

where $\bar{\mathcal{G}}$ and $\mathcal{H}$ are functions of the input matrices $\{\tilde{\mathbf{D}}\}$ that return a single $\tilde{\mathbf{D}}$ and a weight matrix $\mathbf{H}$.

Notice that problem (P – 5) has similarities both to problem (P – 3), in that a weight matrix is included in the formulation, and to (P – 4), in that the solution can be found by fast and efficient Newton methods [13].

Solving problem (P – 5) is, as put by Boyd *et. al* [13], "a matter of technology." In order to reap the benefit of the WLS formulation in a manner similar to that achieved with Chu's LS formulation [11], however, the closed-form expressions of the gradient and Hessian of the objective of (P – 5).

A. Derivation of the Gradient

Consider the cost function given by equation (P – 5),

$$f(\hat{\mathbf{X}}) = \left|\left| F(\hat{\mathbf{X}}) \right|\right|_{\text{F}}^2, \tag{8a}$$

$$F(\hat{\mathbf{X}}) \triangleq \mathbf{H} \circ \left(\left(\mathcal{G}(\{\tilde{\mathbf{D}}\}) \right)^2 - [\langle (\hat{\mathbf{x}}_i - \hat{\mathbf{x}}_j); (\hat{\mathbf{x}}_i - \hat{\mathbf{x}}_j) \rangle^2] \right). \tag{8b}$$

Following similar steps to those in [11], we start form the Fréchet derivative of f with respect to $\hat{\mathbf{X}}$, acting on an arbitrary point $\mathbf{Z} = (\mathbf{z}_1, \cdots, \mathbf{z}_N)$, can be expressed as

$$f'(\hat{\mathbf{X}}) \cdot \mathbf{Z} = \sum_{n=1}^{N} \frac{\partial f}{\partial \hat{\mathbf{x}}_n}(\hat{\mathbf{X}}) \cdot \mathbf{z}_n. \tag{9}$$

The n-th term of the summation on the right-hand side of equation (9) is given by,

$$\frac{\partial f}{\partial \mathbf{x}_n}(\hat{\mathbf{X}}) \cdot \mathbf{z}_n = \langle \frac{\partial}{\partial \hat{\mathbf{x}}_n} F(\hat{\mathbf{X}}) \cdot \mathbf{z}_n; F(\hat{\mathbf{X}}) \rangle. \tag{10}$$

The left term of the inner product shown above is equal to

$$\frac{\partial}{\partial \hat{\mathbf{x}}_n} F(\hat{\mathbf{X}}) \cdot \mathbf{z}_n = -2 \times \tag{11}$$
$$\times \begin{bmatrix} 0 & \cdots & h_{1,n}\langle \mathbf{z}_n; \hat{\mathbf{x}}_n - \hat{\mathbf{x}}_1 \rangle & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ h_{1,n}\langle \mathbf{z}_n; \hat{\mathbf{x}}_n - \hat{\mathbf{x}}_1 \rangle & \cdots & 0 & \cdots & h_{N,n}\langle \mathbf{z}_n; \hat{\mathbf{x}}_n - \hat{\mathbf{x}}_N \rangle \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & h_{N,n}\langle \mathbf{z}_n; \hat{\mathbf{x}}_n - \hat{\mathbf{x}}_N \rangle & \cdots & 0 \end{bmatrix}.$$

Now, observe that the inner product of two matrices $\mathbf{P}$ and $\mathbf{Q}$ is given by

$$\langle \mathbf{P}; \mathbf{Q} \rangle = \sum_{n=1}^{N} \mathbf{p}_n^T \cdot \mathbf{q}_n,$$

where the subscript n denote the n-th column vector.

Let $\mathbf{P} = \frac{\partial f}{\partial \mathbf{x}_n}(\hat{\mathbf{X}}) \cdot \mathbf{z}_n$ and $\mathbf{Q} = F(\hat{\mathbf{X}})$. It follows that

$$\tfrac{\partial f}{\partial \hat{\mathbf{x}}_n}(\hat{\mathbf{X}}) \cdot \mathbf{z}_n = \langle \mathbf{z}_n; -4 \sum_{m=1}^{N} (\hat{\mathbf{x}}_n - \hat{\mathbf{x}}_m) \cdot h_{n,m}^2 v_{n,m} \rangle, \tag{12a}$$

$$v_{n,m} \triangleq \left(g(\{\tilde{\mathbf{D}}\})_{n,m} \right)^2 - \langle \hat{\mathbf{x}}_n - \hat{\mathbf{x}}_m; \hat{\mathbf{x}}_n - \hat{\mathbf{x}}_m \rangle^2. \tag{12b}$$

Comparing equations (12a) and (9) we finally notice that the partial gradient of $f(\hat{\mathbf{X}})$ can be written as

$$\frac{\partial f}{\partial \hat{\mathbf{x}}_n}(\hat{\mathbf{X}}) = -4 \sum_{m=1}^{N} (\hat{\mathbf{x}}_n - \hat{\mathbf{x}}_m) \cdot h_{n,m}^2 v_{n,m}. \tag{13}$$

Therefore, the gradient of f is given by

$$\nabla f(\hat{\mathbf{X}}) = \left[\frac{\partial f}{\partial \hat{\mathbf{x}}_1}(\hat{\mathbf{X}}), \cdots, \frac{\partial f}{\partial \hat{\mathbf{x}}_N}(\hat{\mathbf{X}}) \right]. \tag{14}$$

B. Derivation of the Hessian

The Hessian of the cost function given by equation (8) can also be obtained following similar steps of those in [11]. In particular, the gradient of $f(\hat{\mathbf{X}})$ can be interpreted as an N-fold vector of ε-by-1 blocks. Consequently, the Hessian of $f(\hat{\mathbf{X}})$ is an N-by-N block matrix with ε-by-ε blocks.

Let $\dot{f}_n(\hat{\mathbf{X}})$ denote the partial gradient of $f(\hat{\mathbf{X}})$ with respect to $\hat{\mathbf{x}}_n$,

$$\dot{f}_n(\hat{\mathbf{X}}) \triangleq \frac{\partial f}{\partial \hat{\mathbf{x}}_n}(\hat{\mathbf{X}}). \tag{15}$$

The Jacobian of $\dot{f}_n(\hat{\mathbf{X}})$ is the n-th block-row (n-th row-wise partition of dimension $\varepsilon \times N$) of the Hessian of $f(\hat{\mathbf{X}})$.

The Fréchet derivative of $\dot{f}_n(\hat{\mathbf{X}})$, acting on an arbitrary point $\mathbf{Z}$ is given by the product of the Jacobian of $\dot{f}_n(\hat{\mathbf{X}})$ and $\mathbf{Z}$. Thus,

$$\ddot{f}_n(\hat{\mathbf{X}}) \cdot \mathbf{Z} \triangleq \frac{\partial}{\partial \hat{\mathbf{x}}_n} \dot{f}_n(\hat{\mathbf{X}}) \cdot \mathbf{Z} = \sum_{m=1}^{N} \frac{\partial \dot{f}_n}{\partial \hat{\mathbf{x}}_m}(\hat{\mathbf{X}}) \cdot \mathbf{z}_m. \tag{16}$$

The (n, m)-th block of the Jacobian $\dfrac{\partial \dot{f}_n}{\partial \hat{\mathbf{x}}_m}$ is an ε-by-ε matrix whose product with $\mathbf{z}_m$ is given by

$$\begin{aligned} \frac{\partial \dot{f}_n}{\partial \hat{\mathbf{x}}_m} \cdot \mathbf{z}_m &= -4 \sum_{j=1}^{N} \frac{\partial}{\partial \hat{\mathbf{x}}_m} \left((\hat{\mathbf{x}}_n - \hat{\mathbf{x}}_j) \cdot v_{n,j} \cdot h_{n,j}^2 \right) \cdot \mathbf{z}_m \\ &= \begin{cases} -4 \sum_{j=1}^{N} \left(h_{m,j}^2 \cdot (v_{m,j} \cdot \mathbf{z}_m - 2 \cdot (\hat{\mathbf{x}}_m - \hat{\mathbf{x}}_j) \cdot \langle \mathbf{z}_m; (\hat{\mathbf{x}}_m - \hat{\mathbf{x}}_j) \rangle) \right) & \textit{if } m = n \\ -4 \cdot h_{n,j}^2 \cdot (-v_{n,j} \cdot \mathbf{z}_m - 2 \cdot (\hat{\mathbf{x}}_n - \hat{\mathbf{x}}_j) \cdot \langle -\mathbf{z}_m; (\hat{\mathbf{x}}_m - \hat{\mathbf{x}}_j) \rangle) & \textit{if } m \neq n \end{cases} \end{aligned} \tag{17}$$

Comparing equation (19) and the last form of equation (17), we finally obtain,

$$\begin{aligned} &\frac{\partial \dot{f}_n}{\partial \hat{\mathbf{x}}_m}(\hat{\mathbf{X}}) \\ &= \begin{cases} -4 \sum_{j=1}^{N} \left(h_{m,j}^2 \cdot \left(v_{m,j} \cdot \mathbf{I} - 2 \cdot (\hat{\mathbf{x}}_m - \hat{\mathbf{x}}_j) \cdot (\hat{\mathbf{x}}_m - \hat{\mathbf{x}}_j)^T \right) \right) & \textit{if } m = n \\ -4 \cdot h_{n,j}^2 \cdot (-v_{n,j} \cdot \mathbf{I} + 2 \cdot (\hat{\mathbf{x}}_n - \hat{\mathbf{x}}_j) \cdot (\hat{\mathbf{x}}_n - \hat{\mathbf{x}}_j)^T) & \textit{if } m \neq n \end{cases} \end{aligned} \tag{18}$$

Therefore, the Hessian matrix is given by

$$\nabla^2 f(\hat{\mathbf{X}}) = \left[\frac{\partial \dot{f}_n}{\partial \hat{\mathbf{x}}_m}(\hat{\mathbf{X}}) \right]. \tag{19}$$

Comparing these result with those given in [11], it is found that the closed-form of the gradient and Hessian of the objective of (P – 5), with a weight matrix inserted, does not result in any additional complexity when computing its Hessian, except for the element-wise products with $h_{n,m}$. It is fair to say, therefore, that the complexity of the proposed localization algorithm, based on a weighted least-squares formulation, is the same as that proposed [11]. It will be shown later, however, that the performance of the proposed WLS formulation results in far superior localization performance.

C. The Function $\bar{\mathcal{G}}$ and the Weight Matrix $\mathbf{H}$

We are left with the challenges of designing the function $\bar{\mathcal{G}}$ and computing the weight matrix $\mathbf{H}$. These issues have been tackled in [14] and [15], respectively. Here only a brief review of some of the results thereby is given.

The function $\bar{\mathcal{G}}$ must be designed to return a single $\tilde{\mathbf{D}}$ that best represents the set $\{\tilde{\mathbf{D}}\}$. Several alternatives for $\bar{\mathcal{G}}$ were investigated and compared against one another in terms of complexity and corresponding localization errors [14].

The most straightforward method is to apply MDS to each EDM sample, resulting in a set of "rough" estimates $\hat{\mathbf{X}}_{0:1}, \cdots, \hat{\mathbf{X}}_{0:K}$, which are then averaged into a single estimate $\bar{\hat{\mathbf{X}}}_0$ and used to compute $\tilde{\mathbf{D}}$. A second alternative is the elegant approach of jointly diagonalizing [16] the entire set of Euclidean kernels $\{\tilde{\mathbf{K}}\} = \{\mathcal{K}(\tilde{\mathbf{D}}_1), \cdots, \mathcal{K}(\tilde{\mathbf{D}}_K)\}$ and then reconstruct $\tilde{\mathbf{D}}$ from the resulting kernel. Finally a third alternative, which results in a slightly better performance, is to average the EDM samples as shown below

$$\bar{\mathcal{G}}(\{\tilde{\mathbf{D}}\}) = \bar{\tilde{\mathbf{D}}} = [\bar{\tilde{d}}_{n,m}], \tag{20a}$$

$$\bar{\tilde{d}}_{n,m} \triangleq \begin{cases} \dfrac{1}{K_{n,m}} \displaystyle\sum_{k=1}^{K_{n,m}} \tilde{d}_{n,m:k} & \text{if } K_{n,m} \geqslant 1 \\ 0 & \text{if } K_{n,m} = 0 \end{cases},$$

where $K_{n,m}$ is the number of samples available for $d_{n,m}$.

Amongst the above-mentioned methods, it is found that averaging the EDM samples is both the simplest and best-performing alternative.

In the absence of an optimality theory for the matrix $\mathbf{H}$, we have investigated several solutions for defining a suitable function. In the context of our application, the "rule-of-thumb" conjectured by Wolkowicz [10], namely, that $h_{n,m}$ should be proportional to the relevance of the entry $d_{n,m}$, can be mathematized based on two independent factors. The first is the confidence on the distance estimates $\tilde{d}_{n,m}$, and the second is the impact that the corresponding edge has on the rigidity of the graph representing the network. The form of $\bar{\mathcal{G}}$ given in equation (20) has the additional advantage that it enables a the first of such factors to be quantified based on confidence bounds on the averages $\bar{\tilde{d}}_{n,m}$.

V. Simulation Results and Comparisons

In this section the performance of the metric-MDS, Chu's LS, Wolkovicz's and our proposed WLS algorithms are studied through simulations.

To this end, a network of $A = 8$ anchor nodes (AN) and $N - A = 10$ range-capable nodes (RCN), in a large cube with 10-meter long edges was simulated. The ANs were placed at corners of the cube and have their locations perfectly known, while RCNs are placed at random inside the room (with uniform probability) and are assumed to be capable of measuring the distance between itself and any other node within a range of R_{MAX}, referred to as the *connectivity range*.

The distance estimates are randomly generated with a Gamma distribution, in accordance with the model experimentally derived from the results obtained in a measurement campaign performed in the Centre for Wireless Communications at the University of Oulu, using a pair of impulsive ultra-wideband (UWB) radios. A detailed description of the measurement campaign is reported in [14].

In order to emphasize the relative performance of the techniques considered, isolated from the additional contributions on the method to compute the weight matrix discussed in [15], the weight matrix used in all simulations is simply the connectivity matrix, *i.e.*, $\mathbf{H} = \mathbf{C}$.

The performance of the algorithms is defined in terms of the root-mean-square (rms) positioning error, defined by

$$\zeta^{(\varrho)} \triangleq \left\| [\mathbf{X}]_{\text{UL}:(N-A)\times\eta} - [\hat{\mathbf{X}}^{(\varrho)}]_{\text{UL}:(N-A)\times\eta} \right\|_{\text{F}} / \sqrt{N-A}, \tag{21}$$

where it is assumed that the coordinates of the anchor nodes are placed at the lowest A rows of $\mathbf{X}$.

A. Robustness to Incompleteness

In the first set of simulations, the discussed algorithms are evaluated as function of the completeness alone. To this end, all distances within the connectivity range R_{MAX} were considered perfectly known. This parameter was varied from 2 to 20 meters and for each value of, several random network topologies were tested. The results are displayed in figure 1.

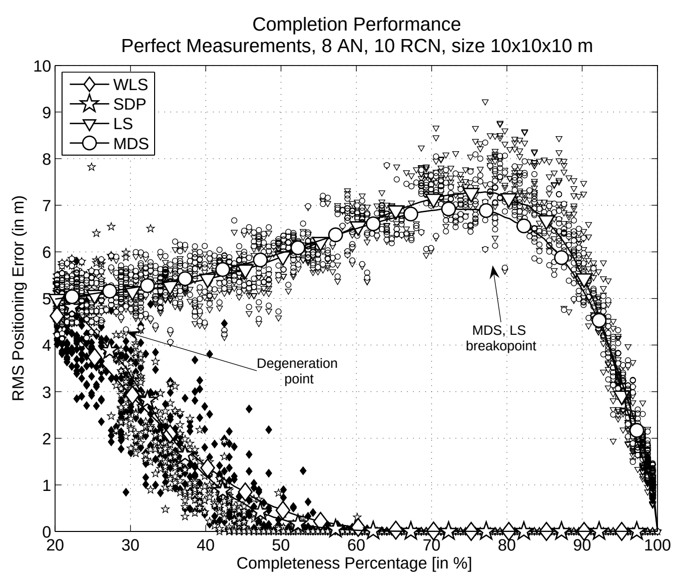


Fig. 1. Performance of localization algorithms from incomplete ranging information.

It is noticeable that the MDS- and the LS-based methods exhibit peaks around the mark of 80% of completion. This mark does not represent an "optimum" operation point, but rather a break point, beyond which both algorithms become ineffective. The convergence to an approximate RMS positioning error of 5 meters is explained by computing, with equation (22), the average RMS positioning error when the entries of $[\mathbf{X}]_{\mathrm{UL}:N-A}$ are uniformly distributed within the interval $[-5, 5]$.

$$\mathrm{E}[\zeta^{(\varrho_{\min})}] = \mathrm{E}\left[\|[\mathbf{X}]_{\mathrm{UL}:N-A\times\varepsilon}\|_{\mathrm{F}}\right]/\sqrt{N-A}, \qquad (22)$$

As for the behavior of the different algorithms within their respective regions of operation, it is found that the metric MDS and Chu's LS algorithm are equivalent and highly sensitive to incompleteness. In comparison, Wolkowicz' and the proposed WLS algorithm are far superior, sustaining up to 50% of incompleteness before degradation increases severely.

B. Robustness to Imperfect Ranging

The SDP and the WLS approaches are compared in the case of fully connected networks with imperfect ranging in figure 2. The plot show that in this case the WLS and the SDP approaches yield identical performance, in terms of both the average and the variance of rms positioning error.

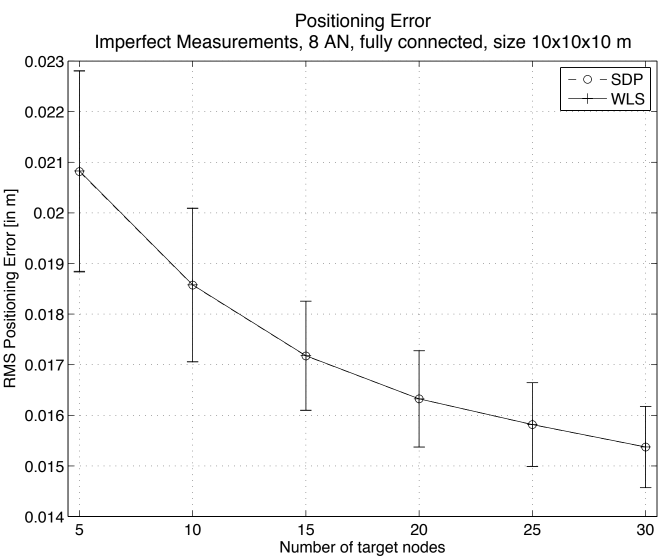


Fig. 2. Performance of WLS- and SDP-based localization algorithms from complete but imperfect EDM samples.

C. Robustness to Imperfect Ranging and Incompletion

Next, a scenario where both imperfect ranging and incomplete connectivity – resulting from either range limitations or an independent activity factor of 80% at each node – are affect the distance estimates is considered. Only the positioning techniques that proved best-performing in the presence of incompleteness, namely the modified Wolkowicz SDP-based method and the proposed WLS algorithm, were compared.

The results are shown in figure 3. It can be seen that Wolkowic's SDP-based method outperforms the proposed WLS algorithm only slightly (at 50% of the trials), at the expense of a much larger computational complexity.

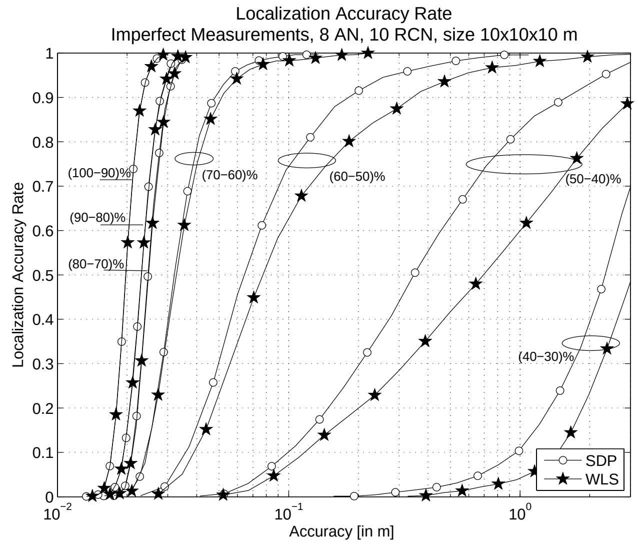
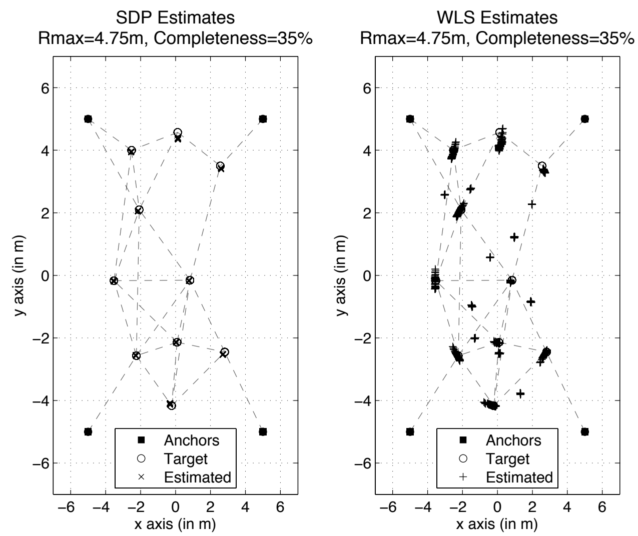
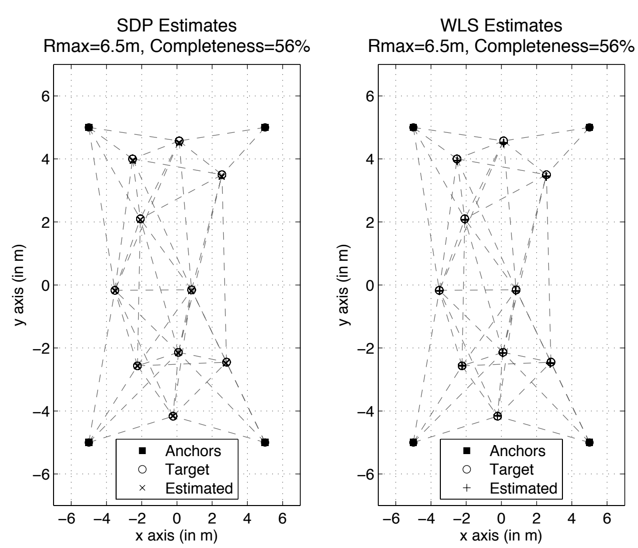


Fig. 3. Performance of localization algorithms from incomplete and imperfect ranging information.



(a) Localization under severe incompleteness.



(b) Localization under moderate incompleteness.

Fig. 4. Snapshots of WLS- and SDP-based position estimates from incomplete EDMs.

It is found that both techniques are able to sustain significant incompleteness while maintaining localization accuracy at the order of centimeters. The figure shows that under severer incompleteness, however, the SDP-based approach is superior to the WLS, albeit at the expense of a much higher complexity. Indeed while the average performances of the proposed and Wolkowicz' algorithms are comparable, as shown in figure figure 1, the SDP solution results in a smaller error variance, especially under severer incompleteness, as indicated by the difference in the success ratios achieved with each technique shown in figure 3.

This is better illustrated, however, in figure 4, which shows snapshots of the position estimates obtained with the proposed WLS-based and Wolkowicz' SDP-based algorithms for different scenarios of completeness EDMs.

D. Complexity Performance

Finally, the SDP-based and the proposed WLS algorithms are compared in terms their complexities, under the same scenario considered in section V-C. To this end, extensive simulations with networks of different topologies and containing 5 to 30 target nodes, were performed and the cpu time required by each algorithm was recorder. The results are shown below in figure 5. The advantage of a minimization based on Newton-method with closed-form of gradient and Hessian is evident.

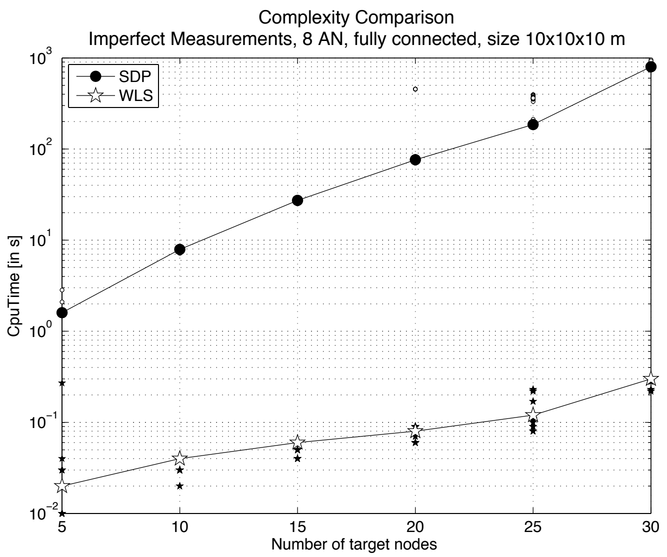


Fig. 5. Complexity comparison between WLS- and SDP-based position estimates from incomplete EDMs.

VI. Conclusions

In this paper, a novel localization technique based on weighted least-squares optimization is proposed, which combines the advantages of the semi-definite programming method of [10], and the least-squares optimization-based algorithm of [11]. It is shown that the proposed solution achieves similar robustness to ranging errors and incompletion and nearly the same average localization accuracy as that of the SDP-based approach [10], with a computational complexity as low as that of the LS-based technique [11], ensured by the fact that closed-form expressions for the gradient and Hessian of the WLS objective, which are derived, is similar to those corresponding to the LS cost-function given in [11].

As such, the proposed WLS technique is a feasible alternative to source localization in sensor networks featuring large numbers of sensors and requiring limited computational power. The ability to handle incomplete and imperfect range information robustly has an impact on the overall power-cost of the application, since it implies that less traffic need be generated for the purpose of collecting intra-node ranging information. The article is a companion to [14], [15], where other related issues such as the form of the weight function used in the objective and the method to combine multiple EDM samples effectively are discussed.

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