GABBA Codes: Generalized Full-Rate Orthogonally Decodable Space-Time Block Codes

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Abstract— **Constellation independent, systematically constructed, orthogonally decodable, full-rate and full-diversity space time block codes (STBCs) generalized to any number of transmit antennas are considered. The proposed codes generalize the ABBA STBC, also known as *quasi*-orthogonal STBC (QO-STBC) and are referred to as generalized ABBA (GABBA) codes. The construction of GABBA codes is systematic in which the encoding matrix for any given number of transmit antennas is obtained by successive operations of a generation and a wrapping functions[1]. GABBA codes admit a linear orthogonal (symbol-by-symbol) decoder which fully decouples all encoded symbols using only linear combinations of the received signals and relying solely on knowledge of the channel at the receiver. The exact bit error probability of the proposed codes in generalized fading channels with the maximum likelihood (ML) receiver is computed and compared against simulation results obtained with the proposed symbol-by-symbol decoder. The comparison reveals that the linear orthogonal decoder achieves nearly the same performance of the ML receiver, despite its remarkably low complexity.**

I. Introduction

The design of space-time codes (STCs) as building blocks of multiple-input-multiple-output (MIMO) systems [1] has been a popular research topic in wireless communications since the discovery of the Alamouti scheme [2] and the orthogonal space-time block-codes (OSTBC) [3].

Although STC techniques were first envisioned for modest spatial-temporal dimensions, the last few years has seen a growing interest on techniques suitable for large/scalable MIMO systems [4]. Arguably, this later development owes to two main factors. The first is the advance of compact multi-mode antennas [5], [6] which, though small in size, can provide large numbers of independent channels by exploiting spatial, pattern and polarization diversities [7]. The second is the increasing popularity of distributed and cooperative systems [8], [9].

These developments contribute to increasing the feasibility of MIMO systems of not only large but, in fact, variable dimensions. For example, in the case of compact multi-mode antennas, MIMO setups of different dimensions may occur because the number of spatial channels available dependents on the scattering properties of the environment. Analogously, in the case of distributed architectures, variable MIMO setups arise naturally since cooperative transmission can only be employed after the cooperating group is assembled [10].

It is known that MIMO techniques can significantly improve both user and rate capacities (efficiencies) of wireless communications systems [11]–[13]. Although space-time transmission diversity techniques in point-to-point links are known to be sub-optimum from a Shannon capacity viewpoint [14], these techniques have their extremely low-complexity to their advantage. This trade-off argument is especially compelling when considered the average throughputs truly achievable by MIMO systems are in practice limited by the use of digital modulation [15]. It is in distributed and cooperative diversity systems, however, that scalable space-time coding techniques have most of their potential [16].

It can therefore be said that modulation-independence, systematic constructibility, full-rateness and orthogonal decodability with performance comparable to that of the maximum likelihood (ML) decoder are essential when considering STBCs for scalable MIMO systems. These requirements make a strong case for linear STBCs. With the Alamouti scheme limited to 2 antennas and alternatives such as the OSTBCs, the QO-STBCs [17]–[19] and the square-embeded STBCs [20] undermined by sub-unitary rateness and/or diversity loss [21], many have taken upon the challenge of designing improved solutions [22]–[25]. To the best of our knowledge, however, these and all other similar contributions presently known fail to simultaneously deliver all the above-mentioned requirements.

This paper deals with a family of space-time block codes that satisfies all these objectives. These codes are a generalization of ABBA STBCs, a.k.a. *quasi*-orthogonal STBC (QO-STBC) [17]–[19], thus called *generalized* ABBA (GABBA) codes. The paper is a companion to [26], where the construction and decoding of the GABBA codes were first presented[2]. Here, the focus is on algorithms and performance analysis. Brief descriptions of both the GABBA encoder and decoder and corresponding Matlab implementations are provided[3]. The exact bit error rate (BER) probability of GABBA codes over PSK and QAM modulated symbols in uncorrelated, memoryless, non-i.i.d fading channels is derived under the assumption of perfect channel knowledge at the receiver and ML decoding. Comparing the theoretical results against computer simulations, it is shown that orthogonally decoded GABBA STBCs achieve nearly the same performance achievable with the (prohibitively complex) ML decoder.

[1]Followed by column puncturing if $n_t \neq 2^n,\ n \in \mathbb{N}$.

[2]The combination of both articles with full details can be found in [27].

II. The GABBA Encoder

Consider the following vector of k symbols taken from an arbitrary complex constellation, to be linearly space-time encoded for rate-one transmission using n_t transmit antennas:

$$\mathbf{s_k} \triangleq [s_1, \cdots, s_k]^{\mathrm{T}}, \tag{1}$$

where $^{\mathrm{T}}$ denotes transpose.

The minimum number k of symbols that can be packed into a fully dense STBC matrix $\mathbf{C}$ of size k–by–n_t such that: a) no symbol repeats a column (full use of the space-diversity); and b) all symbols appear exactly n_t times (full-rateness), is given by [28]:

$$k = 2^{\lceil \log_2 n_t \rceil}, \tag{2}$$

where $\lceil x \rceil$ denotes the largest integer not exceeding x.

Let us define the following auxiliary functions,

$$C_1(\mathbf{X},\mathbf{Y}) = \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ -\mathbf{Y} & \mathbf{X} \end{bmatrix} \tag{3}$$

$$C_2(\mathbf{X},\mathbf{Y}) = \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ -\mathbf{Y}^{\mathrm{H}} & \mathbf{X}^{\mathrm{H}} \end{bmatrix} \tag{4}$$

where $\mathbf{X}$ and $\mathbf{Y}$ are, in general, n–by–n matrices ($n \in \mathbb{N}_+$) of complex entries, and $^{\mathrm{H}}$ denotes transpose conjugate operation.

Square matrices $\mathbf{C_k}$, with k satisfying equation (2), are referred to as *mother* GABBA *matrices*. The k–by–k mother GABBA matrix is obtained as follows:

i) Apply $C_1(\mathbf{X},\mathbf{Y})$ $k-1$ times, first over $\mathbf{s}$, later over the outputs of the previous operation;
ii) Apply $C_2(\mathbf{X},\mathbf{Y})$ over the output of the last stage of i).

Matrices for $n_t \neq 2^m$ are n_t-column partitions of $\mathbf{C_k}$, and are denoted $\mathbf{C}_{n_t}$. The Matlab GABBA encoder given below can be used to generate both symbolic and numeric GABBA encoding matrices of any desired size.

Matlab Implementation of GABBA Encoder

```
function C = GABBAEncoder(s)
% Example 1: C_8 (algebraic)
%   syms s1 s2 s3 s4 s5 s6 s7 s8;
%   s = [s1 s2 s3 s4 s5 s6 s7 s8];
%   C = GABBAEncoder(s);
% Example 2: C_19 (numeric) from C_32
%   s = randn(1,32) + j*randn(1,32);
%   C = GABBAEncoder(s);
%   C(:,[11:16 26:32])=[];
k = length(s); s = reshape(s,[1 1 k]);
while k > 2;
 for n = 1:k/2,
  C(:,:,n)=ABBA1(s(:,:,2*n-1),s(:,:,2*n));
 end
 s = C; clear C; k = k/2;
end
C = ABBA2(s(:,:,1),s(:,:,2));
% --- Auxiliary functions ---
function C=ABBA1(x,y); C=[[x y];[-y x]];
function C=ABBA2(x,y); C=[[x y];[-y' x']];
```

[3] The Matlab codes here provided are aimed at allowing the readers to easily verify our claims, and may not be used for any other purposes without the written, signed and expressed permission of this author.

III. The GABBA Decoder

In this section, the orthogonal decoding algorithm is described. Due to space limitations, the description shall be very brief and limited to highlighting the key steps and features. The reader is referred to [26], [27] for a detailed description and related mathematics of the derivation.

If a k–by–n_t GABBA matrix is used to transmit a vector of arbitrary complex symbols, the received signal vector $\mathbf{r_k}$ at a given receive antenna, in the presence of a block fading channel and additive noise can be written as

$$\mathbf{r_k} = \mathbf{C}_{n_t}\,\mathbf{h_k} + \mathbf{n_k} = \mathbf{H_k}\,\bar{\mathbf{s}}_\mathbf{k} + \mathbf{n_k}, \tag{5}$$

where the *augmented symbol vector* $\bar{\mathbf{s}}_\mathbf{k}$ is given by

$$\bar{\mathbf{s}}_\mathbf{k} = [s_1, \cdots, s_k, s_1^*, \cdots, s_k^*]^{\mathrm{T}}, \tag{6}$$

where * denotes complex conjugate.

From equation (5) it is clear that $\mathbf{H_k}$ is constructed by simply re-arranging[4] the channel entries in an in accordance with the corresponding symbols in the encoding matrix $\mathbf{C}_{n_t}$. It is also evident that $\mathbf{H_k}$ can be written as

$$\mathbf{H_k} = \begin{bmatrix} \mathbf{H}_{\mathbf{k}_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{\mathbf{k}_2} \end{bmatrix}, \tag{7}$$

where $\mathbf{H}_{\mathbf{k}_1}$ and $\mathbf{H}_{\mathbf{k}_2}$ are both $k/2$–by–k matrices.

Analogously to the case of the ABBA code, the *quasi*-orthogonality property of the encoding matrices $\mathbf{C}_{n_t}$ [17]–[19] is preserved in the encoded channel matrices (see [26], [27]), such that $\hat{\mathbf{r}}_\mathbf{k}$ can be split into to vectors each dependent on mutually exclusive $k/2$-tuples of symbols by the following linear combination of the receive vector

$$\hat{\mathbf{r}}_\mathbf{k} = [\mathbf{H_k}^{\mathrm{H}}]^{\mathrm{U}:k}\cdot\mathbf{r_k} + \mathbf{r_k}^*\cdot[\mathbf{H_k}^{\mathrm{T}}]^{\mathrm{R}:k} = \begin{bmatrix} \mathbf{H}_{\mathbf{k}/2}\cdot\mathbf{s}_{\mathbf{k}/2} + \mathbf{n}_{\mathbf{k}/2_1} \\ \mathbf{H}_{\mathbf{k}/2}\cdot\tilde{\mathbf{s}}_{\mathbf{k}/2} + \mathbf{n}_{\mathbf{k}/2_2} \end{bmatrix}. \tag{8}$$

where $\mathbf{n}_{\mathbf{k}/2_1}$ and $\mathbf{n}_{\mathbf{k}/2_2}$ are transformed noise terms and $[\mathbf{X}]^{\mathrm{U}:k}$ and $[\mathbf{X}]^{\mathrm{R}:k}$ denote the upper and right k-partition of $\mathbf{X}$, respectively.

Equation (8) indicates that GABBA codes admit *quasi*-orthogonal decoding, truly extending [17]–[19]. Fortunately, it can furthermore be shown [26], [27] that the *quasi*-orthogonality property of $\mathbf{H_k}$ "propagates" to $\mathbf{H}_{\mathbf{k}/2}$. Specifically, it is found that, for all $1 \leqslant n \leqslant \log_2 K$

$$\boldsymbol{\mathcal{H}}^{n\,\mathrm{T}}_{\frac{K}{2^n}}\cdot\boldsymbol{\mathcal{H}}^{n}_{\frac{K}{2^n}} = \begin{bmatrix} \boldsymbol{\mathcal{H}}_1^n & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\mathcal{H}}_2^n \end{bmatrix}, \tag{9}$$

where

$$\boldsymbol{\mathcal{H}}^n_{K/2^n} = \left[[\mathbf{H}^n_{K/2^n}]^{\boldsymbol{p}_{0:K/2^n}}\,[\mathbf{H}^n_{K/2^n}]^{\boldsymbol{p}_{1:K/2^n}}\right], \tag{10}$$

and $\boldsymbol{p}_{0:N}$ and $\boldsymbol{p}_{1:N}$ are permutation indexes given by

$$\boldsymbol{p}_{0:N} = \{1,\cdots,i,\cdots\}, \text{ such that } p_N(i)=0, (i \leqslant N), \tag{11}$$

$$\boldsymbol{p}_{1:N} = \{2,\cdots,j,\cdots\}, \text{ such that } p_N(j)=1, (j \leqslant N), \tag{12}$$

where

$$p_N(x) = \mathrm{mod}\left(x\cdot\lfloor\log_2 N\rfloor - x + \sum_{n=1}^{\lfloor\log_2 N\rfloor-1}\left\lfloor\frac{x}{2^n}\right\rfloor, 2\right), \tag{13}$$

with $\lfloor x \rfloor$ denoting the largest integer no greater than x and $\mathrm{mod}(x,2)$ denoting the rest of division by 2.

[4] Hence, we shall hereafter refer to $\mathbf{H_k}$ as the *encoded channel matrix*.

It is evident that $\mathcal{H}_1^n$ and $\mathcal{H}_2^n$ both reduce to complex scalars at $n = \log_2 K$, yielding all symbols to be completely fully decoupled from nested linear combinations of $\mathbf{r_k}$. Alternatively, the nested linear combinations can be performed algebraically, yielding a single-step orthogonal decoder for the GABBA codes, which can be described by

$$\hat{\mathbf{s}}_\mathbf{k} = \mathbf{F}^{\mathrm{H}}_{\mathbf{k}:1} \cdot \mathbf{r_k} + \mathbf{r_k}^* \cdot \mathbf{F}^{\mathrm{T}}_{\mathbf{k}:2}, \tag{14}$$

where $\mathbf{F}_{\mathbf{k}:1}$ and $\mathbf{F}_{\mathbf{k}:2}$ are functions of the channel vector, uniquely determined for each GABBA code.

MATLAB IMPLEMENTATION OF GABBA DECODER

```
function s_hat = GABBADecoder(rr,hh)
% Inputs: rr = received vector/matrix
%         hh = ch. estimate vector/matrix
% Output: s_hat = soft symbol estimates
[K,nr] = size(rr); r_hat = zeros(K/2,2);
H = zeros(K/2,K/2,2);
for a = 1:nr,
 r = rr(:,a);  h = hh(:,a);
 % -------- First Stage ---------
 Hk = GABBAEncodedChannelMatrix(h);
 for n = 1:2,
  r_hat(:,n) = r_hat(:,n) + ...
   Hk(:,1 + (n-1)*K/2 : n*K/2)'*r + ...
   (r'*Hk(:,1+K+(n-1)*K/2 : K+n*K/2)).';
 end
 H1 = Hk(1:K/2,1:K);
 H2 = Hk((1+K/2):K,(1+K):2*K);
 H(:,:,1) = H(:,:,1)+...
           conj((H1*H1'+H2*H2')/2);
end
H(:,:,2)=H(:,:,1); NormH=norm(H(:,:,1));
H = H/NormH; r_hat = r_hat/NormH;
% ------- Higher Stages --------
Ps = [[1:K/2].' [(1:K/2) + K/2].'];
if K > 2,
 [p1,p2] = PermutationIndexes(1:K/2);
 for i = 1:(log2(K)-1),
  r=r_hat; r_hat=[]; Ps_hat=Ps; Ps=[];
  for n = 1:2^i,
   Ps=[Ps Ps_hat(p1(1:K/(2^(i+1))),n)...
       Ps_hat(p2(1:K/(2^(i+1))),n)];
   v = H(:,:,mod(n,2)+1).'*r(:,n);
   r_hat=[r_hat v(p1(1:K/(2^(i+1))))...
         v(p2(1:K/(2^(i+1))))];
  end
  H_hat = H(:,:,1).'*H(:,:,2); H = [];
  H(:,:,1) = H_hat(p1(1:K/(2^(i+1))),...
                p1(1:K/(2^(i+1))));
  H(:,:,2) = H_hat(p2(1:K/(2^(i+1))),...
                p2(1:K/(2^(i+1))));
  NormH = norm(H(:,:,1)); H = H/NormH;
  r_hat = r_hat/NormH;
 end
end
for n = 1:K,
 s_hat(Ps(n),1) = ...
    r_hat(n)/H(:,:,mod(n+1,2)+1);
end
% --- Auxiliary Function 1 ---
function [p0,p1] = PermutationIndexes(k)
% Input: vector of indexes k = 1,2,...K
N = length(k); p = k;
for n = 1:ceil(log2(N))-1,
 p = p + floor((k-1)/(2^n));
end
p=mod(p,2); p0=find(p); p1=find(p-1);
% --- Auxiliary Function 2 ---
function H=GABBAEncodedChannelMatrix(h,K)
K = length(h); % Must be a power of 2
H1 = reshape(h,[1 1 K]);
H2 = reshape([h(K/2+1:end)...
              h(1:K/2)],[1 1 K]);
while K >= 2;
 for n = 1:K/2,
  C1(:,:,n) = ...
        ABBA1(H1(:,:,2*n-1),H1(:,:,2*n));
  C2(:,:,n) = ...
        ABBA2(H2(:,:,2*n-1),H2(:,:,2*n));
 end
 H1 = C1; H2 = C2; clear C1 C2; K = K/2;
end
K = length(h);
H = [H1(1:K/2,:) zeros(K/2,K);...
     zeros(K/2,K) H2(1:K/2,:)];
% --- Auxiliary functions 3 & 4 ---
function C=ABBA1(x,y); C=[[x y];[y -x]];
function C=ABBA2(x,y); C=[[x -y];[y x]];
```

IV. PERFORMANCE OF GABBA CODES

In this section, the performance of GABBA codes is studied both analytically and through simulations. Given the difficulty in computing the coefficients $\mathbf{F}_{\mathbf{k}:1}$ and $\mathbf{F}_{\mathbf{k}:2}$ in equation (14) explicitly, the exact bit error probability (BEP) of orthogonally decoded GABBA codes is hard to determine in a generalized and fashion. Therefore, our approach is to derive analytical expressions corresponding to the ML decoder (which can be seen as bounds), and compare these against simulation results obtained using the proposed GABBA orthogonal decoder.

A. Exact BEP of GABBA Codes with ML Decoding

The BEP of digitally modulated signals in a fading channel can be computed by averaging the corresponding error probability in the additive white Gaussian noise (AWGN) channel over the statistics of the fading process [29]. The performance of a ML-decoded full-rate full-diversity STBC is equivalent to that of a $(n_t \times n_r)$-branch maximum ratio combiner with the receive power equal to n_r [30].

Let the average symbol-energy-to-noise-power ratio (Es/No) of the n-th pair of transmit-to-receive antennas be denoted by γ_n, and the envelope fading process corresponding to each diversity branch be described by the continuous probability density function (PDF) $p_{\Gamma_n}(\gamma_n)$. Furthermore, let $P_b(\gamma_n)$ denote the BEP of the digital modulation system over the n-th branch under AWGN conditions. Then, the BEP of the full-rate full-diversity GABBA code is given by

$$\bar{P}_b(\gamma) = \underbrace{\int_0^\infty \cdots \int_0^\infty}_{(n_t \times n_r)-\text{fold}} \prod_{n=1}^{n_t \times n_r} P_b(\gamma_n) \cdot p_{\Gamma_n}(\gamma_n)\; d_{\gamma_1} \cdots d_{\gamma_{n_t n_r}}. \quad (15)$$

1) M-ary PSK Modulation:

The BEP of M-ary PSK modulated signals over the AWGN channel were (believed to be) calculated exactly by Lee in [31]. A minor error (for higher-order constellations) was recently discovered by Lassing *et al.* and corrected in [32], [33]. In particular, from [32, eqs. (2) and (3)], we have

$$P_{b:\text{PSK}}(\gamma|M) = \frac{1}{\log_2 M} \sum_{k=1}^{M-1} \bar{d}_{k:\text{PSK}} \cdot P_{k:\text{PSK}}(\gamma|M), \quad (16)$$

with

$$\bar{d}_{k:\text{PSK}} = 2\left|\frac{k}{M} + \left\lfloor \frac{k}{M} \right\rceil\right| + 2 \sum_{i=2}^{\log_2 M > 2} \left|\frac{k}{2^i} + \left\lfloor \frac{k}{2^i} \right\rceil\right|, \quad (17)$$

where $\lfloor x \rceil$ rounds x to its nearest integer, and $P_{k:\text{PSK}}(\gamma|M)$ is the probability that a received M-ary PSK symbol falls k sectors away from the sector it belongs to due to the effect of AWGN, under the signal-to-noise ratio γ.

A convenient formula for $P_{k:\text{PSK}}(\gamma|M)$ can be derived from [34, eq. (24)], yielding[5]

$$P_{k:\text{PSK}}(\gamma|M) = \frac{1}{2\pi}\left[\int_0^{\pi(1-\delta_k^-)} e^{-\gamma \cdot \frac{\sin^2(\pi\delta_k^-)}{\sin^2(\theta)}} d\theta - \int_0^{\pi(1-\delta_k^+)} e^{-\gamma \cdot \frac{\sin^2(\pi\delta_k^+)}{\sin^2(\theta)}} d\theta\right], \quad (18)$$

where $\delta_k^- = (2k-1)/M$ and $\delta_k^+ = (2k+1)/M$ for uniform M-ary PSK constellations.

The exact average BER probability of M-ary PSK over fading channels can then be fond from equations (15) through (18), using the moment generating function (MGF) method (see [29]). Before the resulting formula is given, however, notice that: a) if a STBC of rate ρ is used, each symbol is transmitted $1/\rho$ times by each transmit antenna (so that the energy of the transmit constellation increases proportionally); and b) if the diversity order of the code is $\eta < n_t$, the lower diversity order can be translated as a uniform loss in the diversity contributions of all branches. These observations which leads to the more general formula given below

$$\bar{P}_{b:\text{PSK}}(\gamma|M,\rho,\eta) = \frac{1}{2\log_2 M} \sum_{k=1}^{M-1} \bar{d}_k \times \quad (19)$$
$$\times\Big(\tilde{I}(\delta_k^-, g_{\text{PSK}}(\delta_k^-), n_t \cdot n_r|\rho,\eta) - \tilde{I}(\delta_k^+, g_{\text{PSK}}(\delta_k^+), n_t \cdot n_r|\rho,\eta)\Big),$$

with $g_{\text{PSK}}(\delta)$ and $\tilde{I}(\delta, g, N|\rho,\eta)$ respectively given by

$$g_{\text{PSK}}(\delta) = \sin^2(\pi\delta), \quad (20)$$

$$\tilde{I}(\delta, g, N|\rho,\eta) = \frac{1}{\pi} \cdot \int_0^{\pi(1-\delta)} \left(\prod_{n=1}^{N} \mu_{\gamma_n}\left(\frac{-g}{\rho \cdot \sin^2(\theta)}\right)\right)^{\eta} d\theta, \quad (21)$$

where $\mu_{\gamma_n}(x)$ is the MGF of the PDF $p_{\Gamma_n}(\gamma_n)$.

2) M-ary QAM Modulation:

The exact BEP for QAM modulation in the AWGN channel was derived by Cho *et al.* [35]. In particular, for regular M-ary QAM constellations we have [35, eq. (16)],

$$P_{b:\text{QAM}}(\gamma|M) = \frac{1}{\log_2 \sqrt{M}} \sum_{k=1}^{\log_2 \sqrt{M}} P_{k:\text{QAM}}(\gamma|M). \quad (22)$$

In equation (22), $P_{k:\text{QAM}}(\gamma|M)$ is the probability that the k-th bit in the M-ary symbol is in error, which can be put into the more convenient form shown below [35, eq. (14)]

$$P_{k:\text{QAM}}(\gamma|M) = \frac{2}{\pi\sqrt{M}} \times \sum_{i=0}^{\left(\sqrt{M} - \sqrt{\frac{M}{4^k}}\right)-1} d_{i:\text{QAM}} \cdot \int_0^{\frac{\pi}{2}} e^{\frac{-2\gamma(2i+1)^2}{(M-1)\sin^2(\theta)}} d\theta \quad (23)$$

where

$$d_{i:\text{QAM}} = (-1)^{\left\lfloor \frac{i \cdot 2^{k-1}}{\sqrt{M}} \right\rfloor} \left(2^{k-1} - \left\lfloor \frac{i \cdot 2^{k-1}}{\sqrt{M}} + \frac{1}{2} \right\rfloor\right). \quad (24)$$

The similarities between equations (16) and (22), and between equations (18) and (23) are evident. Substituting equations (22) through (24) into equation (15), invoking the MGF method and the arguments in the paragraph preceding equation (19), we finally obtain

$$\bar{P}_{b:\text{QAM}}(\gamma|M,\rho,\eta) = \frac{2}{\sqrt{M}\log_2 \sqrt{M}} \times \quad (25)$$
$$\sum_{k=1}^{\log_2 \sqrt{M}} \left[\sum_{i=0}^{(1-2^{-k})\sqrt{M}-1} d_{i:\text{QAM}} \cdot \tilde{I}\left(\frac{1}{2}, g_{\text{QAM}}(i), n_t \cdot n_r|\rho,\eta\right)\right].$$

where

$$g_{\text{QAM}}(i) = 3 \cdot (2i+1)^2/(2M-2). \quad (26)$$

B. Orthogonally Decoded GABBA Codes

In this section, the performance of orthogonally decoded GABBA codes are compared to the exact formulas (bounds) derived above. First, in figure 1, the BEP of a large GABBA code ($n_t = 64$) in independent and identically distributed (i.i.d.) Rayleigh fading channels considered. The simulated results (white markers) were obtained with the orthogonal, symbol-by-symbol decoder described in section III, while the analytical results (black markers) were computed using equation (25), with $\rho = 1$ and $\eta = 1$. The remarkably low-complexity of the scheme is emphatically illustrated in this figure. Indeed, given the number of antennas and the modulation utilized, simulated results with the ML-decoder are virtually impossible to obtain at a complexity order of $\mathcal{O}(64^{63})$ per decoded symbol. In contrast, the complexity of the orthogonal decoder is only $\mathcal{O}(64)$ per symbol estimate. Despite the much lower complexity, it is found that the performance attained by the GABBA STBC with the orthogonal decoder is close to that theoretically achievable with the ML decoder. Figure 1 also provides evidence that the GABBA orthogonal decoder is not a mere zero-forcing technique, since the apparent SNR loss observed at the simulated results decreases with the number of receive antennas.

[5] Equation (18) appears (with a minor error) in [29, pp. 201, eq. (129)]

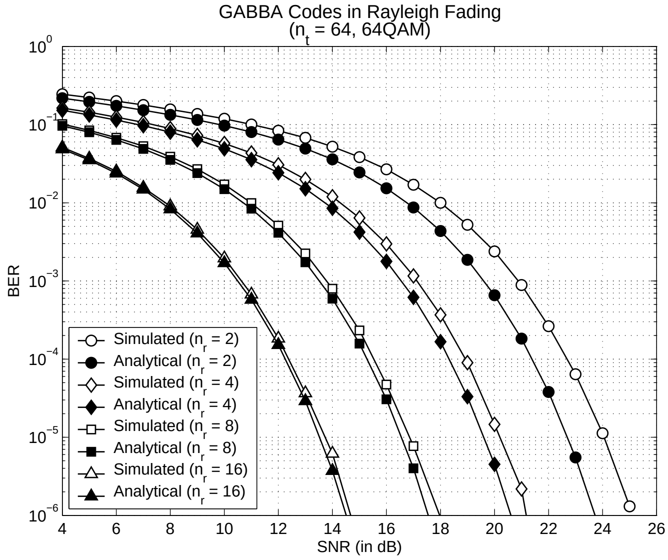

Fig. 1. Performance of the GABBA code $\mathbf{C}_{64}$ over 64-QAM modulation in Rayleigh block-fading channels with various numbers of receive antennas.

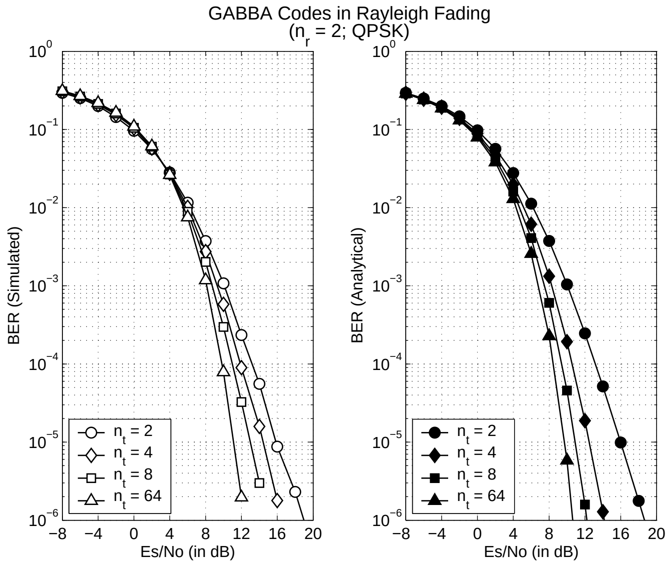

Fig. 2. Performance of GABBA codes of different sizes over QPSK.

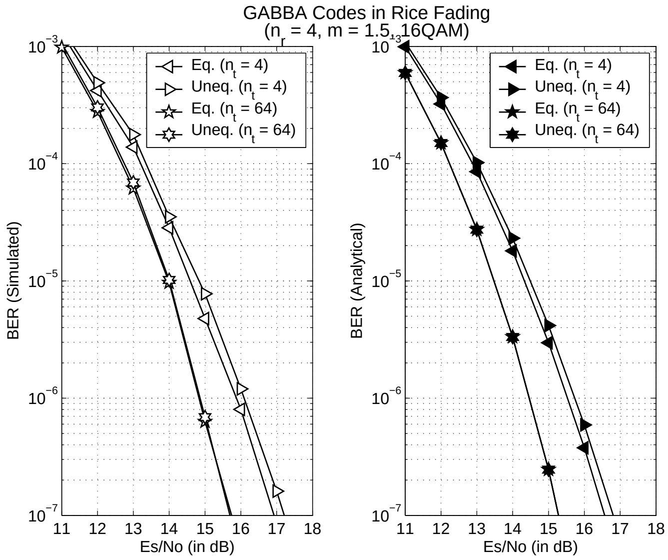

Fig. 3. Performance of GABBA codes of different sizes over 16QAM symbols in Rice block-fading channels with 4 receive antennas. Curves for both equal and unequal (linear) average power distribution profiles across the transmit antennas are given.

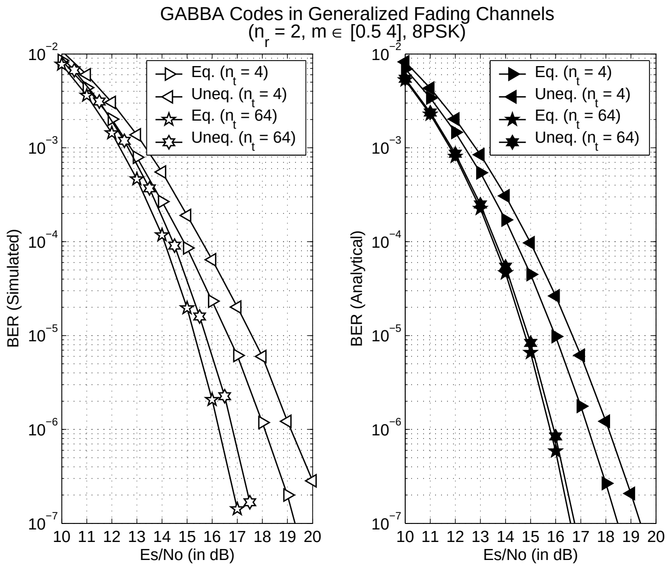

Fig. 4. Performance of GABBA codes of different sizes over 8PSK symbols in generalized block-fading channels with 2 receive antennas. Curves for scenarios with linear power distribution and fading severity factors m spanning the interval $[0.5, 4]$, as well as equipower channels and a mean fading severity factor $m = 2.5$ are given.

This claim is further supported by the results shown in figure 2, where the simulated and analytical BER probabilities of GABBA codes of various sizes over QPSK modulated signals in the Rayleigh fading channel are compared. It is found that even with a small number of receive antennas, the performance of orthogonally decoded GABBA codes is close to the theoretical bound.

Next, we turn our attention to the performance of large GABBA codes in the presence of non-i.i.d. fading channels. First, consider the case of channels with the same fading statistics but different powers [36], [37]. This case is associated to a cooperative scenario [9] with GABBA-encoded transmission, where the multiple cooperating transmitters are in the same environment (same statistics), but at different locations (different average powers).

The cooperation is assumed to be perfect, *i.e.*, cooperating transmitters are perfectly synchronized and coordinated, such that their transmit signals arrive at the receiver at the same time, at every transmission epoch. It is also assumed that the average power of each transmit channel is an independent random uniformly distributed in the interval $[0, P_{\max}]$. In this case, it can be shown [27] using order statistics that the power distribution that best describes the average scenario faced by the system is

$$\bar{\wp}_k = k \cdot P_{\max}/(K+1). \tag{27}$$

where $\wp_k$ denotes the power of the ordered branches.

It ia seen in figure 3 that, as expected, the impact of unequal power distribution on the ergodic performance of GABBA-encoded systems diminishes for large codes.

Finally, the performances of GABBA codes in uncorrelated block-fading channels with equal and unequal statistics are compared. The results, obtained both analytically and through computer simulations, are shown in figure 4. The curves corresponding to the unequal fading channels were obtained by assigning to the k-th branch a fading severity factor m_k and a power factor Ω_k such that m_k spans the interval $[0.5, 4]$ uniformly, with $m_{k+1} > m_k$, while $\sum \Omega_k = 1$, with $\Omega_{k+1} < \Omega_k$. In other words, the channels experiencing more severe fading are also the most powerful ones, such that no particular channel in the ensemble is "dominant". This selection of parameters is associated to scenario faced by peer-to-peer MIMO systems employing diversity-optimized multi-mode adaptive antennas with orthogonal pattern adaptability. The comparison shows how the use of a large GABBA code results in more robustness against shadowing and other effects that may affect the power distribution and fading statistics of the diversity branches available in peer-to-peer MIMO systems with multi-mode antennas.

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