$$b^n_{m,k+1}[p] = b^n_{m,k}[p] - b^n_{m,k}[p-1] = \sqrt{\frac{6}{\pi(n+1)(m^2-1)}} \times \left(\left[\exp\left(\frac{6x^2}{(n+1)(1-m^2)}\right)\right]^{(k)} \left(p - \frac{k+(n+1)}{2}\right) - \left[\exp\left(\frac{6x^2}{(n+1)(1-m^2)}\right)\right]^{(k)} \left(p - 1 - \frac{k+(n+1)}{2}\right)\right)$$
$$= \sqrt{\frac{6}{\pi(n+1)(m^2-1)}} \frac{12\alpha}{(n+1)(1-m^2)} \times \left[\exp\left(\frac{6x^2}{(n+1)(1-m^2)}\right)\right]^{(k+1)} (\alpha)$$
$$\text{for at least one } \alpha \in \left[p - 1 - \frac{k+n+1}{2}, p - \frac{k+n+1}{2}\right]$$
$$\approx \sqrt{\frac{6}{\pi(n+1)(m^2-1)}} \times \left[\exp\left(-\frac{6x^2}{(n+1)(m^2-1)}\right)\right]^{(k+1)} \times \left(p - \frac{k+1+n}{2}\right).$$

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Closed-Form Correlation Functions of Generalized Hermite Wavelets

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***Abstract*—A closed-form expression is given for the correlation functions of generalized Hermite wavelets, constructed from an also-generalized definition of Hermite polynomials. Due to their Gaussianity, these wavelets can be used as a tool in the analysis or design of systems involving nonsinusoidal wavelets as well as to model impulsive waveforms found in real-world applications and signal processing problems. As such, the formula is potentially applicable to various areas of science.**

***Index Terms*—Correlation functions, Hermite expansions, Hermite wavelets.**

I. Introduction

Due to its Gaussianity, Hermite wavelets (which are often referred to as Hermite functions in the mathematics literature) have proved to be an increasingly important tool in signal processing, with application to various areas of science ranging from image coding and compression [1], to biomedical engineering [2], to neural networks [3], to wireless communications [4]–[6], to high-definition radar systems [7].

Several fundamental issues in such applications depend on the correlation properties of the set of wavelets employed. Hermite wavelets are constructed from the Hermite polynomials through multiplication by an exponential decay term, such that different choices of decay coefficient and Hermite polynomials result in different wavelet sets. At least two different definitions of Hermite polynomials are widely encountered in the mathematics literature [8]–[11], and the Hermite wavelets constructed from either of these definitions are the most commonly found in engineering applications [1]–[4], [12], [13].

This is, in part, because these particular Hermite wavelets have the particular characteristic of forming complete (density-one) sets of orthonormal functions but also because the correlation functions of these

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sets can be easily found in the literature [3], [5], [6], [14], whereas the correlation properties of generalized Hermite wavelets are not known.

With respect to the orthogonality of Hermite wavelets, it can be said that although density-one orthonormal sets may the natural choice when orthonormality is the only fundamental property desired, there are cases when the context of the application requires additional properties that may lead to the choice of lower density orthonormal Hermite sets. It is known, for instance, that antennas act as differentiators over ultra-wideband signals [15], [16], such that the signals generated and received by such devices are related by double differentiation.[1] From the signal processing and implementation points of view, it is therefore advantageous, in ultra-wideband systems, to employ a set of Hermite wavelets that is closed under differentiation (such that the derivative of any waveform in the set is also in the set). In [5], it was shown that, in fact, such wavelets (which are a particular case of the generalized construction discussed here) form a density-half orthonormal set (where even and odd functions are mutually orthogonal), which removes the need for different waveform generators at transmission and reception, enables multiple waveforms to be generated by successive differentiation of the Gaussian pulse (without scaling or time-basis adjustments), and may even result in performance improvement (if a vector-based detection scheme is employed).

Based on this example, it is easy to imagine that many other special constructions of Hermite wavelets may find applications in various problems involving the use of Gaussian waveforms. In this paper, a closed-form expression is given for the correlation functions of such waveforms.

II. Generalized Hermite Polynomials and Wavelets

Consider a generalized definition of Hermite polynomials given by the following Rodrigues' formula

$$H_n(t;a) \triangleq (-1)^n e^{\frac{t^2}{a}} \frac{d^n}{dt^n} e^{-\frac{t^2}{a}} \tag{1}$$

where $n \in \mathbb{N}$ is the order of the polynomial, and $a \in \mathbb{R}(a \neq 0)$.

The two families of Hermite polynomials commonly found in the literature result if $a = 1$ or $a = 2$ is substituted into (1) (see [8, p. 691] or [9, p. 56]).

The following recursive formulae for generalized Hermite polynomials can be easily derived directly from (1).

$$H_0(t;a) = 1 \tag{2a}$$
$$H_1(t;a) = \frac{2t}{a} \tag{2b}$$
$$H_{n+1}(t;a) = \frac{2t}{a} H_n(t;a) - \frac{2n}{a} H_{n-1}(t;a). \tag{2c}$$

Generalized Hermite wavelets are constructed from (1) as follows:

$$\psi_n(t;a,b) \triangleq \frac{H_n(t;a) e^{\frac{-t^2}{b}}}{\mathcal{N}_n(a,b)} (b \in \mathbb{R}; b > 0). \tag{3}$$

The first problem encountered with such a generalized definition of Hermite wavelets is that a closed-form expression for the normalization factors $\mathcal{N}_n(a,b)$ is not known, such that explicit formulae for the waveforms are not available.

At this point, the following results, obtained by inspection, are introduced.

[1]There are indications that the propagation channel may act as an integrator [17], such that this relationship may reduce to a single differentiation.

Proposition 1: The generalized Hermite polynomial can be written in closed form as

$$H_n(t;a) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} c_n(k;a) t^{n-2k} \tag{4}$$

where $\lfloor x \rfloor$ denotes the largest integer lower or equal to x, and $c_n(k;a)$ is given by

$$c_n(k;a) = (-1)^k 2^{n-2k} \left(\frac{n!}{(n-2k)!k!} \right) a^{k-n}. \tag{5}$$

Proposition 2: An explicit product formula for generalized Hermite polynomials is given by

$$H_n(t;a) H_m(t;a) = \sum_{k=0}^{n} k! \left(\frac{2}{a}\right)^k \binom{n}{k} \binom{m}{k} H_{n+m-2k}(t;a) \tag{6}$$

where $n \leq m$, without loss of generality.

Although these results were obtained by inspection, one can easily verify them to be true by direct comparison with the outcomes of (1) [or using (2a) through (2c)] and their products. We also emphasize the similarity of (4) and (6) to equivalent formulae known for the special case of $a = 1$ (see, for instance, [3, eq. (10)] and [10, p. 532]).

Using the results of Propositions 1 and 2, the normalization factors of generalized Hermite wavelets can be calculated in closed form. Starting from the definition of normality, i.e., that the energy of $\psi_n(t;a,b)$ is unitary, we arrive at

$$\mathcal{N}_n(a,b)^2 = \int_{-\infty}^{\infty} H_n(t;a)^2 e^{\frac{-2t^2}{b}} dt = \sqrt{\frac{\pi b}{2}} E[H_n(t;a)^2]_{\frac{\sqrt{b}}{2}} \tag{7}$$

where $E[f(t)]_\sigma$ denotes the expected value of $f(t)$ as if t were a Gaussian random of zero mean and standard deviation σ, as hinted in [14].

Combining (4)–(7), we obtain

$$\mathcal{N}_n(a,b)^2 = \sqrt{\frac{\pi b}{2}} \sum_{k=0}^{n} k! \left(\frac{2}{a}\right)^k \binom{n}{k}^2 \times \sum_{p=0}^{n-k} c_p(2n-2k;a) E\left[t^{2(n-k-p)}\right]_{\frac{\sqrt{b}}{2}}. \tag{8}$$

The terms $E[t^m]_\sigma$ in (8) are the central moments of the Gaussian random t of zero mean and standard deviation σ, which are given by [18]

$$E[t^m]_\sigma = \begin{cases} 0, & \text{for } m \text{ odd} \\ \sigma^m \prod_{k=1}^{\frac{m}{2}} (2k-1), & \text{for } m \text{ even.} \end{cases} \tag{9}$$

Substituting $\sigma = (\sqrt{b}/2)$ and replacing the product in (9) by a more convenient form [8], we have

$$E[t^m]_{\frac{\sqrt{b}}{2}} = \begin{cases} 0, & \text{for } m \text{ odd} \\ \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \left(\frac{b}{2}\right)^{\frac{m}{2}} = \left\langle \frac{1}{2} \right\rangle_{\frac{m}{2}} \left(\frac{b}{2}\right)^{\frac{m}{2}}, & \text{for } m \text{ even.} \end{cases} \tag{10}$$

where $\langle x \rangle_m$ denotes the mth-order Pochhammer's symbol on x [8].

Let us define the following auxiliary function, which will be proven to be very convenient (especially in the next section):

$$\xi(x,y) \triangleq \begin{cases} 0, & \text{for } x \text{ odd} \\ \left\langle \frac{1}{2} \right\rangle_{\frac{x}{2}} y^{\frac{x}{2}}, & \text{for } x \text{ even.} \end{cases} \quad (11)$$

Noting that $n - k - p \in \mathbb{N}$, such that all the exponents of t in (8) are even, we finally obtain, from (8), (10), and (11), the following relationship:

$$\mathcal{N}_n(a,b)^2 = \sqrt{\frac{\pi b}{2}} \sum_{k=0}^{n} k \left(\frac{2}{a}\right)^k \binom{n}{k}^2 \times \sum_{p=0}^{n-k} c_p(2(n-k);a)\,\xi\left(2(n-k-p), \frac{b}{2}\right). \quad (12)$$

Fig. 1 illustrates how impulsive waveforms found in real-world applications (such as in the generation of pulses for ultra-wideband communications and radar) can be modeled using generalized Hermite wavelets to a higher degree of precision than that achievable with "conventional" Hermite wavelets [8]. This is possible because the parameters a and b can be set to different values, resulting in waveforms of slightly different shapes.

III. Correlation Formulas

We are now ready to compute an explicit formula for the correlation functions of generalized Hermite wavelets. By definition we have

$$R_{n,m}(\tau; a_n, b_n, a_m, b_m) \triangleq \int_{-\infty}^{\infty} \psi_n(t; a_n, b_n)\psi_m(t-\tau; a_m, b_m)\,dt. \quad (13)$$

Making a change of variables from t to $t+\eta$, substituting (3) into (13), and computing η such that the exponential terms on t and τ can be separated,[2] we obtain

$$R_{n,m}(\tau; a_n, b_n, a_m, b_m) = \frac{e^{-\frac{\tau^2}{b_n+b_m}}}{\mathcal{N}_n(a_n,b_n)\mathcal{N}_m(a_m,b_m)} \int_{-\infty}^{\infty} H_n\left(t + \frac{b_n}{b_n+b_m}\tau; a_n\right) \times H_m\left(t - \frac{b_m}{b_n+b_m}\tau; a_m\right) e^{-\frac{b_n+b_m}{b_n b_m}\tau^2}\,dt. \quad (14)$$

Again, note that the above integral is in fact the expected value of the product of Hermite polynomials, where t is a Gaussian random of zero mean and standard deviation $\sigma = \sqrt{(b_n b_m)/(2(b_n+b_m))}$. Thus,

$$R_{n,m}(\tau; a_n, b_n, a_m, b_m) = \frac{e^{-\frac{\tau^2}{b_n+b_m}}}{\mathcal{N}_n(a_n,b_n)\mathcal{N}_m(a_m,b_m)} \sqrt{\frac{\pi b_n b_m}{b_n+b_m}} \times E\left[H_n\left(t + \frac{b_n}{b_n+b_m}\tau; a_n\right) \times H_m\left(t - \frac{b_m}{b_n+b_m}\tau; a_m\right)\right]_{\sqrt{\frac{b_n b_m}{2(b_n+b_m)}}}. \quad (15)$$

[2]This gives $\eta = (b_n)/(b_n+b_m)\tau$.

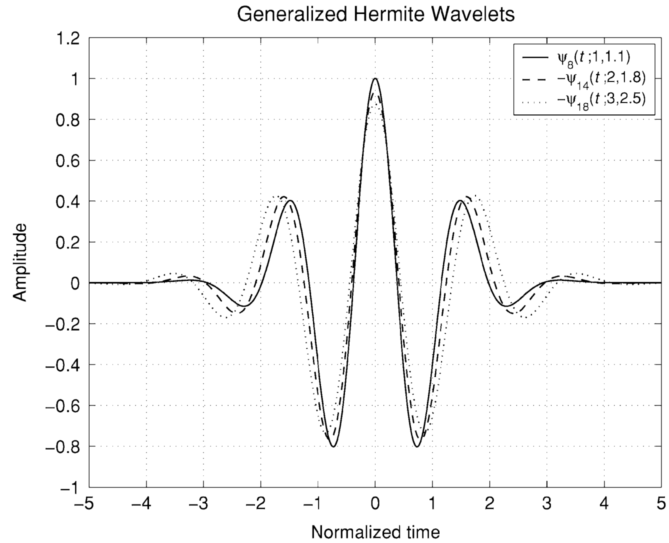

Fig. 1. Examples of generalized Hermite wavelets with different parameters.

Substituting (4) into (15), we obtain

$$R_{n,m}(\tau; a_n, b_n, a_m, b_m) = \frac{e^{-\frac{\tau^2}{b_n+b_m}}}{\mathcal{N}_n(a_n,b_n)\mathcal{N}_m(a_m,b_m)} \sqrt{\frac{\pi b_n b_m}{b_n+b_m}} \times \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{i=0}^{n-2k} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{j=0}^{m-2q} (-1)^j d_n(k,i;a_n) d_m(q,j;a_m) \times \left(\frac{b_n}{b_n+b_m}\tau\right)^i \left(\frac{b_m}{b_n+b_m}\tau\right)^j \times E\left[t^{n+m-2(k+q)-(i+j)}\right]_{\sqrt{\frac{b_n b_m}{2(b_n+b_m)}}} \quad (16)$$

where

$$d_n(k,i;a) = \binom{n-2k}{i} c_n(k;a) \quad (17)$$

and where the binomial theorem below was used (see [8])

$$(t+\eta)^n = \sum_{k=0}^{n} \binom{n}{k} t^{n-k}\eta^k. \quad (18)$$

Using (11) in (16), we finally arrive at the following explicit formula for the correlation functions of generalized Hermite wavelets

$$R_{n,m}(\tau; a_n, b_n, a_m, b_m) = \frac{e^{-\frac{\tau^2}{b_n+b_m}}}{\mathcal{N}_n(a_n,b_n)\mathcal{N}_m(a_m,b_m)} \sqrt{\frac{\pi b_n b_m}{b_n+b_m}} \times \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{i=0}^{n-2k} \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{j=0}^{m-2q} (-1)^j d_n(k,i;a_n) d_m(q,j;a_m) \times \left(\frac{b_n}{b_n+b_m}\tau\right)^i \left(\frac{b_m}{b_n+b_m}\tau\right)^j \times \xi\left(n+m-2(k+q)-(i+j), \frac{b_n b_m}{b_n+b_m}\right). \quad (19)$$

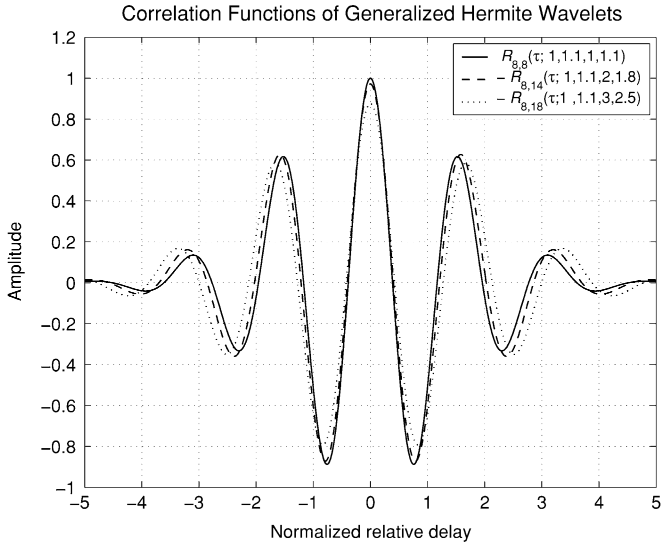

Fig. 2. Few correlation functions involving the generalized Hermite wavelets shown in Fig. 1.

Note that the result is completely general and allows for the parameters a_n, b_n a_m, and b_m to be all different, which essentially implies that the formula can be used to compute the correlation between generalized Hermite wavelets of different families. If, for instance, generalized Hermite wavelets are used to model waveforms found in real-world applications (such as impulsive signals used in radar and wideband communications [7], [15]), this feature may be useful in the analysis of the performance of such systems in the presence of waveform imperfections or distortions, such as those resulting from instabilities in the oscillatory circuits used in the waveform generators.

A few plots of (19), showing the correlation properties of some pairs of the wavelets of Fig. 1, are given in Fig. 2. The figure strongly suggests that the correlation functions of generalized Hermite wavelets can also be represented by generalized Hermite wavelets. If this indication is proven generally true,[3] an immediate and important consequence would be a significant simplification of (19). Further investigation of this issue is currently being carried out by the author.

IV. Conclusion

In this correspondence, a closed-form expression for the correlation function of generalized Hermite wavelets, defined over generalized Hermite polynomials, was presented.

Due to its flexibility, generalized Hermite wavelets can be used to increase the accuracy of mathematical models for Gaussian-like impulsive waveforms commonly found in the physical world. These wavelets can also be used to construct orthonormal sets of waveforms with properties not found in the conventional set of Hermite function (such as closure to differentiation), which can be required by some applications.

In light of its generality, the result is potentially applicable to various areas of science, relying on the use of nonsinusoidal impulsive waveforms for analytical or design purposes, such as image coding and compression, biomedical engineering, neural networks, wireless communications, and radars.

[3]At this stage, one can resort to a numerical approach by finding the pair of parameters a and b that best describes the result obtained with the closed-form expression provided.

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