Arbitrarily Tight Upper and Lower Bounds on the Gaussian Q-function and Related Functions

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***Abstract*—We present a new family of tight lower and upper bounds on the Gaussian Q-function $Q(x)$. It is first shown that, for any x, the integrand $\varphi(\theta;x)$ of the Craig representation of $Q(x)$ can be partitioned into a pair of complementary convex and concave segments. As a consequence of this property, integrals of $\varphi(\theta;x)$ over arbitrary intervals within its convex region can be lower-bounded by Jensen's inequality and upper-bounded by Cotes' quadrature rule, with the opposite occurring for the concave region $\varphi(\theta;x)$. The combination of these complementary bounds yield a complete family of both lower and upper bounds on $Q(x)$, which are expressed in terms of elementary transcendental functions and can be made arbitrarily tight by finer segmentation. A by-product of the method is that various other functions, such as the squared Gaussian Q-function $Q^2(x)$, the 2D joint Gaussian Q-function $Q(x,y,\rho)$, and the generalized Marcum Q-function $Q_M(x,y)$, can also be both upper and lower bounded with arbitrarily tightness, which to the best of our knowledge finds no precedence in the literature. Explicit examples of the latter applications are given.**

I. INTRODUCTION

The Gaussian Q-function $Q(x)$, a variation of the complementary error function $\mathrm{erfc}(x)$ popular in the communication theory literature, is frequently found in the analysis of multi-antenna communication systems over fading channels [1], [2]. Due to its importance, the efficient evaluation of $Q(x)$ has been a subject of concern of many authors in the area [3]–[7].

In particular, the discovery several years ago of an alternative representation of $Q(x)$ due to Craig [8], that enables its evaluation via an integral with finite integration limits, has inspired an enormous amount of activity around the matter of further simplifying, bounding and approximating $Q(x)$ and related functions. This effort led to, amongst other results [2], the discovery of convenient forms for functions such as powers of $Q(x)$ [9], the 2D joint Gaussian Q-function $Q(x,y;\rho)$ [10]–[12], the first-order Marcum Q-function $Q_1(x,y)$ [13], the generalized Marcum Q-function $Q_M(x,y)$ [14], and the Nuttall Q-function $Q_{M,N}(x,y)$ [15]–[17].

The literature established connections between several of these important functions and the more fundamental Gaussian Q-function, which further strengthens the importance of mathematically tractable, computationally efficient and accurate bounds on the latter. In this article, new bounds on the Gaussian Q-function $Q(x)$ are derived with basis on the convexity-concavity partitionability property of the integrand $\varphi(\theta;x)$ of Craig's representation of $Q(x)$. Such a partitionability property, which is established here, enables the application of Jensen's inequality to lower(upper) bound integrals of $\varphi(\theta;x)$ over arbitrary intervals within its convex(concave) region.

Complementarily, the convexity-concavity partitionability of $\varphi(\theta;x)$ also ensures that the same integrals are upper(lower)-bounded by their numerical evaluations via Cotes' quadrature (trapezoid) rule [18]. The combination of these Jensen and Cotes bounds yield a complete family of both lower and upper bounds on $Q(x)$, which can be made arbitrarily tight by finer segmentation.

The proposed Jensen-Cotes (JC) bounds are expressed in terms of irrational, trigonometric, inverse-trigonometric and exponential functions only, and admit further simplification via elementary algebra and well-known inequalities, which ensures easy implementation in standard computer programming languages and tools (such as C and Matlab) and fast evaluation.

In addition to arbitrary tightness and computational efficiency, a very significant outcome of the proposed bounding method is that lower bounds of equivalent quality to the upper-bounding counter-parts are obtained, a fact that to the best of our knowledge finds no precedent in the related literature. Yet another significant by-product of the method is that various functions related to the Q-function can also be upper- and lower-bounded likewise, examples of which are given.

A brief summary of the remainder of the article is as follows. In section II, Lemmas on the convex-concave separability of $\varphi(\theta;x)$ and on the construction of Jensen and Cotes bounds on the integrals of corresponding segments are stated and proved, followed by Theorem 1 which describes the JC-bounds on $Q(x)$. In section III, results obtained by direct application of Theorem 1 are given, along with some additional discussions and comparisons with the best exponential bounds currently known, as given in [19]. A few examples on the application of the Theorem 1 to the derivation of bounds on function related to $Q(x)$ are discussed in section IV, and concluding remarks are offered in section V.

II. JENSEN-COTES BOUNDS ON $Q(x)$

Consider the integral

$$Q(x|a,b) \triangleq \frac{1}{\pi}\cdot\int_a^b \varphi(\theta;x)\,d\theta, \tag{1}$$

where

$$\varphi(\theta;x) \triangleq \exp\left(\frac{-x^2}{2\sin^2(\theta)}\right) \tag{2}$$

Equation (1), with integration limits $a=0$ and $b=\pi/2$ and $x \geqslant 0$ is the well-known Guassian Q-function [8], which appears in many communications theory problems [1], [2].

Lemma 1 - Convex-concave partitionability of $\varphi(\theta;x)$:
For all $x \geqslant 0$, $\varphi(\theta;x)$ is convex for $0 \leqslant \theta \leqslant \bar{\theta}_x$ and concave for $\bar{\theta}_x \leqslant \theta \leqslant \pi/2$, where

$$\bar{\theta}_x = \operatorname{asin}\left(\frac{1}{2}\cdot\sqrt{(x^2+3)-\sqrt{(x^2-1)^2+8}}\right). \tag{3}$$

Proof: In what follows, $0 \leqslant \theta \leqslant \pi/2$ and $x \geqslant 0$ unless otherwise specified[1]. Consider the derivatives

$$\dot{\varphi}(\theta;x) = \frac{d\varphi(\theta;x)}{d\theta} = x^2\frac{\cos\theta}{\sin^3\theta}\varphi(\theta;x) \geqslant 0, \tag{4}$$

$$\begin{aligned}\ddot{\varphi}(\theta;x) &= \frac{d^2\varphi(\theta;x)}{d\theta^2} = \\ &= \frac{x^2}{\sin^6\theta}\left[2\sin^4\theta-(3+x^2)\sin^2\theta+x^2\right]\varphi(\theta;x).\end{aligned} \tag{5}$$

Solving $\ddot{\varphi}(\theta;x)=0$ for θ is equivalent to finding the roots of the expression between brackets in equation (5). Letting $t=\sin^2(\theta)$ leads to the following quadratic polynomial on t, with coefficients parameterized by x

$$2\,t^2-(3+x^2)\,t+x^2=0. \tag{6}$$

The roots of this polynomial are

$$t_1 = \tfrac{1}{4}(x^2+3+\sqrt{(x^2-1)^2+8}), \tag{7}$$

$$t_2 = \tfrac{1}{4}(x^2+3-\sqrt{(x^2-1)^2+8}). \tag{8}$$

Clearly $t_1 \geqslant 1$ and therefore leads to solutions of equation (5) lying outside the domain of θ, thus inadmissible. On the other hand, $0 \leqslant t_2 < 1$, which yields, for each x, the unique inflection point of $\varphi(\theta;x)$ – hereafter referred to as *critical angle* – given by equation (3). Finally, since the polynomial in equation (6), and consequently $\ddot{\varphi}(\theta;x)$, are non-negative for $t \leqslant t_2$, it follows that $\varphi(0 \leqslant \theta \leqslant \bar{\theta}_x;x)$ is convex, while $\varphi(\bar{\theta}_x \leqslant \theta \leqslant \pi/2;x)$ is concave, concluding the proof. ■

Lemma 2 - Arbitrarily tight Jensen bounds on $Q(x|a,b)$:
Let $\varphi(\theta;x)$ be convex for all $\theta \in [a,b]$. Then,

$$Q(x|a,b) \geqslant Q_{\mathrm{J}}(x|a,b,N), \tag{9}$$

where $Q_{\mathrm{J}}(x|a,b,N)$ is referred to as the N-term Jensen bound, defined by

$$Q_{\mathrm{J}}(x|a,b,N) \triangleq \frac{(b-a)}{\pi N}\sum_{n=1}^{N}\varphi\Big(a+\tfrac{(2n-1)}{2N}\cdot(b-a);x\Big). \tag{10}$$

Conversely, if $\varphi(\theta;x)$ is concave in $\theta \in [a,b]$,

$$Q(x|a,b) \leqslant Q_{\mathrm{J}}(x|a,b,N). \tag{11}$$

Proof: Suffice it to prove inequality (9), since the proof for inequality (11) is equivalent. In our context, the Jensen inequality states that for any continuous interval Θ of the real line over which $g(\theta)$ is a real-valued measurable function, $\psi(\theta)$ is a convex function over the range of $g(\theta)$, and $f(\theta)$ is a non-negative function such that $\int_\Theta f(\theta)\,d\theta = 1$, we have

$$\int_\Theta \psi(g(\theta))f(\theta)\,d\theta \geqslant \psi\left(\int_\Theta g(\theta)f(\theta)\,d\theta\right), \tag{12}$$

with equality holding if and only if (iff) $\psi(\theta)=\alpha\,\theta+\beta$.

Next, consider the uniform partition of the interval $[a,b]$ into N subintervals $\Theta_n \triangleq \langle a+\frac{n-1}{N}(b-a), a+\frac{n}{N}(b-a)\rangle$, where the chevron brackets may denote either open or closed boundaries. For simplicity, we shall also write $\Theta_n = \langle a_n, b_n\rangle$.

Let $\psi(\theta)=\varphi(\theta;x)$, $g(\theta)=\theta$ and $f_n(\theta)=\frac{1}{b_n-a_n}$, such that the aforementioned conditions are satisfied. Then,

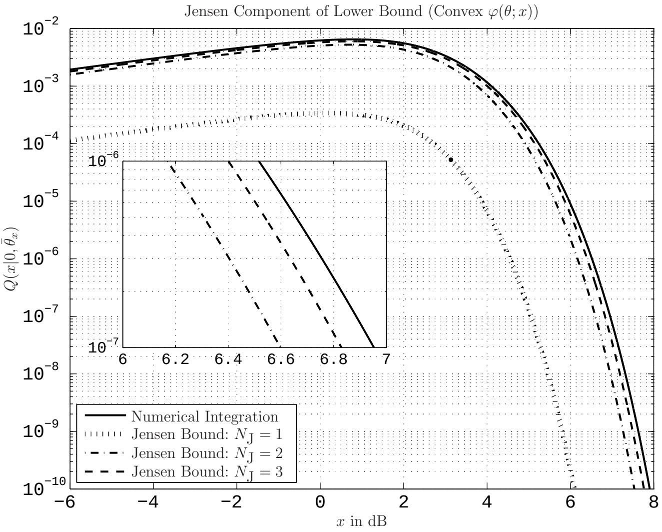
$$\frac{1}{b_n-a_n}\int_{a_n}^{b_n}\varphi(\theta;x)\,d\theta \geqslant \varphi\left(\tfrac{1}{b_n-a_n}\int_{a_n}^{b_n}\theta\,d\theta;x\right)=\varphi\left(\tfrac{b_n+a_n}{2};x\right). \tag{13}$$

Using inequality (13) into equation (1) and summing over n yields the inequality (9).

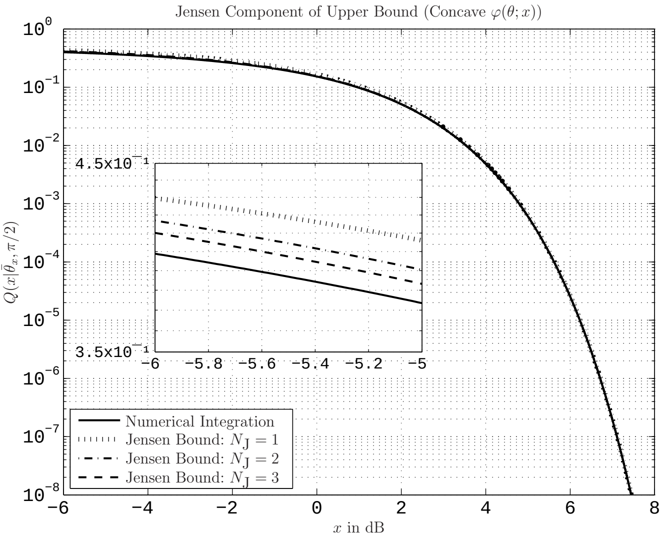
To prove arbitrary tightness, define $\vartheta_n \triangleq a+\frac{(2n-1)}{2N}\cdot(b-a)$ and notice that $a_n \leqslant \vartheta_n \leqslant b_n$, and that $\frac{b-a}{N}=(b_n-a_n)$. Then, straight from equation (10) we obtain

$$\lim_{N\to\infty}Q_{\mathrm{J}}(x|a,b,N) = \frac{1}{\pi}\lim_{N\to\infty}\sum_{n=1}^{N}\varphi(\vartheta_n;x)\cdot(b_n-a_n) = Q(x|a,b), \tag{15}$$

where the sum after the first equality sign is readily identified as a Riemann sum, and the last equality follows its asymptotic exactness in the limit $N\to\infty$. ■



(a) Inequality (9).



(b) Inequality (11).

Fig. 1. Jensen bounds on $Q(x|0,\theta_x)$ and $Q(x|\theta_x,\pi/2)$.

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[1] This is with no loss of generality since $Q(-x)=1-Q(x)$.

In order to illustrate the results of Lemma 2, examples of lower and upper Jensen bounds achieved over the portions of $Q(x)$ corresponding to convex and concave regions of $\varphi(\theta;x)$ – namely, $Q(x|0,\theta_x)$ and $Q(x|\theta_x,\pi/2)$, respectively – with 1, 2 and 3 terms are shown in figure 1. The plots indicate that the Jensen bounds (particularly the upper bounds) are indeed very tight even when only a few terms are used.

Lemma 3 - Arbitrarily tight Cotes bounds on $Q(x|a,b)$: Let $\varphi(\theta;x)$ be concave for all $\theta\in[a,b]$. Then,

$$Q(x|a,b)\geqslant Q_{\text{C}}(x|a,b,N),\tag{16}$$

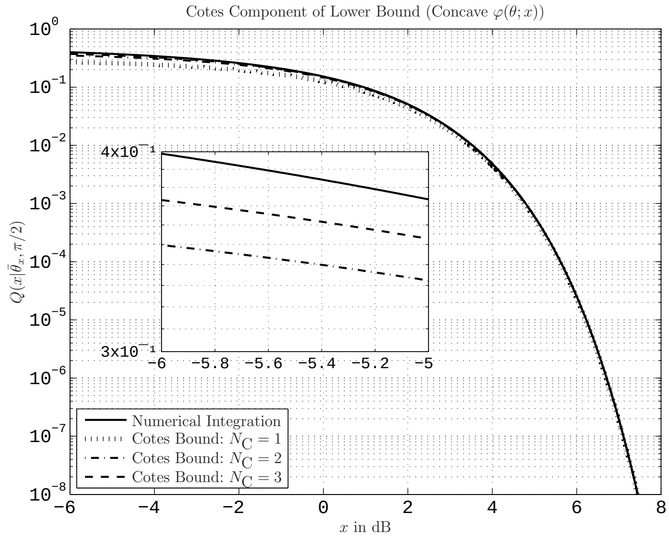
where

$$Q_{\text{C}}(x|a{,}b{,}N)\triangleq\frac{(b-a)}{\pi N}\left[\frac{\varphi(b;x)+\varphi(a;x)}{2}+\sum_{n>0}^{N-1}\varphi\Big(a+\frac{n}{N}(b-a);x\Big)\right]\tag{17}$$

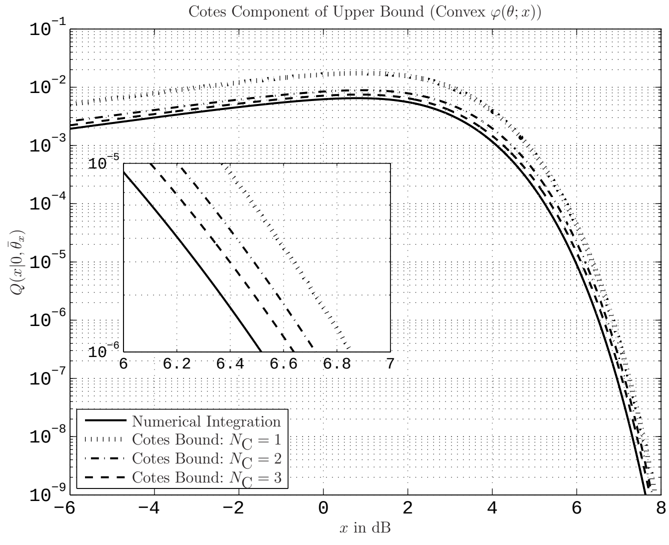
and the notation is intentionally modified so as to emphasize that the summation over n is performed iff $N>1$.

Conversely, if $\varphi(\theta;x)$ is convex in $[a,b]$,

$$Q(x|a,b)\leqslant Q_{\text{C}}(x|a,b,N).\tag{18}$$



(a) Inequality (16).



(b) Inequality (18).

Fig. 2. Cotes bounds on $Q(x|0,\theta_x)$ and $Q(x|\theta_x,\pi/2)$.

Proof: Again, suffice it to prove the first inequality, the second being similarly proved. Clearly $\varphi(\theta;x)$ in $[a,b]$ is lower-bounded by the straight lines connecting the points $(a_n,\varphi(a_n;x))$ and $(b_n,\varphi(b_n;x))$, *i.e.*,

$$\varphi(\theta;x)\geqslant\alpha_n\cdot\theta+\beta_n,\theta\in\langle a_n,b_n\rangle,\tag{19}$$

where

$$\alpha_n=\tfrac{1}{b_n-a_n}\cdot[\varphi(b_n;x)-\varphi(a_n;x)],\tag{20}$$

$$\beta_n=\tfrac{1}{b_n-a_n}\cdot[b_n\cdot\varphi(a_n;x)-a_n\cdot\varphi(b_n;x)].\tag{21}$$

Substituting inequality (19) into equation (1) leads to inequality (16), and since the this procedure is essentially the composite trapezoidal rule for numerical integration, which is an extension of the 2-point closed Newton-Cotes quadrature formula [18], arbitrary tightness with $N\rightarrow\infty$ is ensured. ■

Figure 2 is a counter-part of figure 1, illustrating Cotes lower and upper bounds on $Q(x|\theta_x,\pi/2)$ and $Q(x|0,\theta_x)$, respectively. Again, it is found that these bounds are quite tight even for N small. Comparing figures 1 and 2, it is seen that Cotes' bounds are tighter where Jensen's are less so, and vice-versa. This complementariness is behind the tight and well-behaved Jensen-Cotes bounds on $Q(x)$, to be shown below.

Theorem 1 - Arbitrarily tight JC-bounds on $Q(x)$: For arbitrary $(N_{\text{J}},N_{\text{C}})\in\mathbb{N}^+$,

$$Q(x)\geqslant Q_{\text{JC}}(x{,}N_{\text{J}}{,}N_{\text{C}})\triangleq Q_{\text{J}}(x|0{,}\bar{\theta}_x{,}N_{\text{J}})+Q_{\text{C}}(x|\bar{\theta}_x{,}\tfrac{\pi}{2}{,}N_{\text{C}}),\tag{22}$$

$$Q(x)\leqslant Q_{\text{CJ}}(x{,}N_{\text{C}}{,}N_{\text{J}})\triangleq Q_{\text{C}}(x|0{,}\bar{\theta}_x{,}N_{\text{C}})+Q_{\text{J}}(x|\bar{\theta}_x{,}\tfrac{\pi}{2}{,}N_{\text{J}}).\tag{23}$$

Proof: First, rewrite

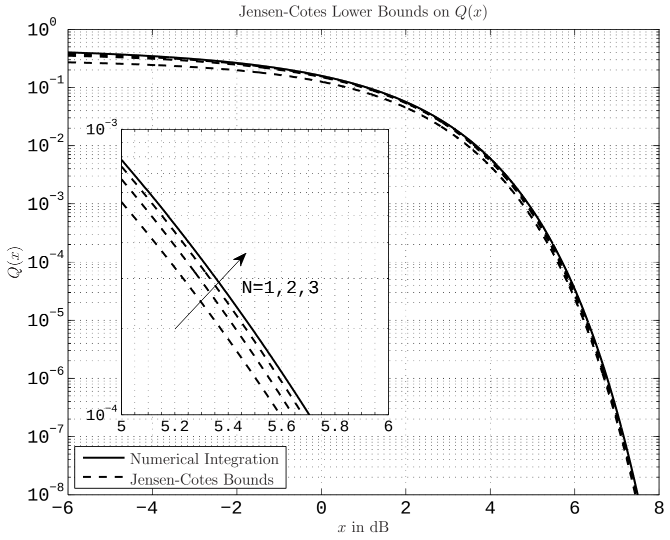
$$Q(x)=Q(x|0,\bar{\theta}_x)+Q(x|\bar{\theta}_x,\pi/2).\tag{24}$$

Next, add inequality (9) with $(a=0,b=\bar{\theta}_x)$ to inequality (16) with $(a=\bar{\theta}_x,b=\pi/2)$, to obtain a complete lower bound on $Q(x)$, by force of Lemmas 1 to 3 and equation (24). Likewise, a complete upper bound on $Q(x)$ results from the sum of inequality (18) with $(a=0,b=\bar{\theta}_x)$ and inequality (11) with $(a=\bar{\theta}_x,b=\pi/2)$. Finally, the arbitrary tightness of the Jensen and Cotes bounds themselves, together with equation (24), ensure the arbitrary tightness of inequalities (22) and (23) with $(N_{\text{C}},N_{\text{J}})\rightarrow\infty$, which concludes the proof. ■

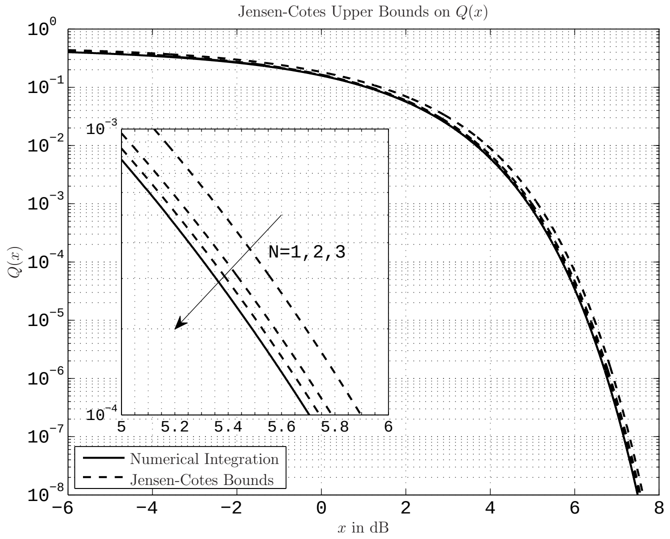
III. RESULTS ON $Q(x)$ AND DISCUSSIONS

The bounds yielded by Theorem 1 involve only elementary transcendental functions (exponential, irrational, trigonometric and inverse trigonometric functions). Though not as manipulable as some found in the literature (most noticeably the exponential bounds given in [19]), the proposed Jensen-Cotes bounds are still very easy to program in standard computer languages and very fast to evaluate, since no numerical integrations or complicated special functions are required. Furthermore, to the best of our knowledge, the proposed Jensen-Cotes lower bounds are the only family of arbitrarily tight lower bounds on $Q(x)$ so far discovered.

Plots of $Q(x)$ and its lower and upper JC-bounds with $N=N_{\text{J}}=N_{\text{C}}=\{1,2,3\}$ are shown in figure 3. The plots show that the proposed bounds are tight over the entire SNR region, and equally good, in fact complementary in that the lower bound is tighter at large x, while the upper bound is tighter at smaller x.



(a) Inequality (22).



(b) Inequality (23).

Fig. 3. Jensen-Cotes lower and upper bounds on $Q(x)$, $N = \{1, 2, 3\}$.

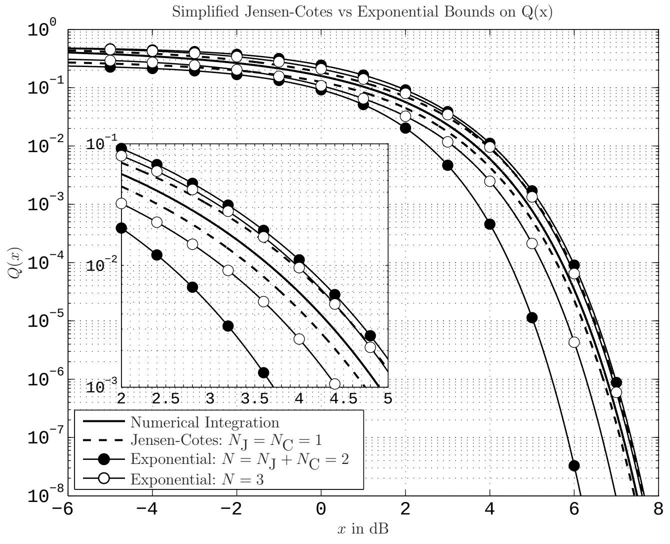


Fig. 4. Comparison of the 2-term exponential upper bound on $Q(x)$ from [19, Eq.(10)] against simplified Jensen-Cotes lower and upper bounds given by inequalities (25) and (26), $N_C = N_J = 1$.

The JC-bounds in their general (arbitrarily-tight) form given in Theorem 1, are adequate for accurate evaluation of $Q(x)$, offering over approximations such as those proposed in [3]–[7] the advantage of certain majorization and minorization of $Q(x)$ at any point, which approximate forms cannot provide. The bounds can also be reduced to simpler forms at negligible sacrifice of tightness. In particular, it can be shown that

$$Q(x)\geqslant\frac{3f_1(x)}{4\pi+\pi f_2(x)}\cdot\exp\left(\frac{-2x^2}{2-f_2(x)}\right)+ \tag{25}$$

$$+\frac{4+(\pi-2)f_2(x)-2f_1(x)}{16+4(\pi-2)f_2(x)}\cdot\left[\exp\left(\frac{-x^2}{2}\right)+\exp\left(\frac{-2x^2}{f_1^2(x)}\right)\right],$$

$$Q(x)\leqslant\frac{f_1(x)}{8+2(\pi-2)f_2(x)}\cdot\exp\left(\frac{-2\,x^2}{f_1^2(x)}\right)+ \tag{26}$$

$$+\frac{4\pi+f_2(x)\pi-6f_1(x)}{8\pi+2\pi f_2(x)}\cdot\exp\left(\frac{-2\,x^2}{2+f_1(x)}\right),$$

where

$$f_1(x)\triangleq\sqrt{(x^2+3)-\sqrt{(x^2-1)^2+8}}, \tag{27}$$

$$f_2(x)\triangleq\sqrt{\sqrt{(x^2-1)^2+8}-(x^2-1)}=\sqrt{4-f_1^2}, \tag{28}$$

and we have used the Shafer-Malěsević's inequalities [20]

$$\frac{3x}{2+\sqrt{1-x^2}}\leqslant\operatorname{asin}(x)\leqslant\frac{\frac{\pi}{\pi-2}x}{\frac{2}{\pi-2}+\sqrt{1-x^2}},\, x\in[0,1]. \tag{29}$$

A comparison of inequalities (25) and (26) against the 2-term exponential upper bound given by [19, Eq.(10)] is shown in figure 4. The figure confirms that the simplified 2-term Jensen-Cotes upper bound outperforms the 2-term exponential upper bound (although the latter is simpler). Once again, the simplified Jensen-Cotes lower bound (which contains 3 exponential terms) is comparatively as tight overall.

IV. APPLICATION TO RELATED FUNCTIONS

Theorem 1 can be applied to the derivation of lower and upper bounds on other functions that can be expressed in terms of variations of the integral given in equation (1), which includes but exceeds erfc-based functions. A few examples of such applications are given below.

Corollary1.1 - Arbitrarily tight JC-bounds on $Q^2(x)$:

$$Q^2(x)\geqslant\begin{cases}Q_{\rm J}(x|0,\bar{\theta}_x,N_{\rm J})+Q_{\rm C}(x|\bar{\theta}_x,\frac{\pi}{4},N_{\rm C}) & \text{for } 0\leqslant x<\sqrt{2}\\ Q_{\rm J}(x|0,\frac{\pi}{4},N_{\rm J}+N_{\rm C}) & \text{for } x\geqslant\sqrt{2}\end{cases} \tag{30}$$

$$Q^2(x)\leqslant\begin{cases}Q_{\rm C}(x|0,\bar{\theta}_x,N_{\rm C})+Q_{\rm J}(x|\bar{\theta}_x,\frac{\pi}{4},N_{\rm J}) & \text{for } 0\leqslant x<\sqrt{2}\\ Q_{\rm C}(x|0,\frac{\pi}{4},N_{\rm C}+N_{\rm J}) & \text{for } x\geqslant\sqrt{2}\end{cases} \tag{31}$$

Proof: Consider the relation [21, Eq. (80)]

$$Q(x)^2=Q(x|0,\pi/4). \tag{32}$$

It is trivial to show directly from equation (3) that $\bar{\theta}_x > \pi/4\ \forall\ x>\sqrt{2}$, such that $\varphi(\theta;x)$ becomes convex in $[0,\pi/4]$ when $x\geqslant\sqrt{2}$. This, together with equation (32) and a revision of Theorem 1 and preceding Lemmas in view of the integration interval $\theta\in[0,\pi/4]$, leads straightforwardly to inequalities (30) and (31). ■

Corollary 1.2 - Arbitrarily tight JC-bounds on $Q(x,y;\rho)$: The 2D Joint Gaussian Q-function $Q(x,y;\rho)$ admits the arbitrarily tight bounds

$$Q(x,y;\rho)\geqslant\tfrac{1}{2}\left[Q_{\text{L:2D}}(x;\tfrac{x}{y},\rho,N_\text{J},N_\text{C})+Q_{\text{L:2D}}(y;\tfrac{y}{x},\rho,N_\text{J},N_\text{C})\right], \quad (33)$$

$$Q(x,y;\rho)\leqslant\tfrac{1}{2}\left[Q_{\text{U:2D}}(x;\tfrac{x}{y},\rho,N_\text{J},N_\text{C})+Q_{\text{U:2D}}(y;\tfrac{y}{x},\rho,N_\text{J},N_\text{C})\right], \quad (34)$$

where[2] $x,y\geqslant 0$ and

$$\begin{aligned}Q_{\text{L:2D}}(u;z,\rho,N_\text{J},N_\text{C})\triangleq{}&Q_\text{J}(u|0,\min(\bar{\theta}_u,\theta^*_{z,\rho}),\lceil\tfrac{N_\text{J}}{2}\rceil+i_1N_\text{C}+i_2\lfloor\tfrac{N_\text{J}}{2}\rfloor)\\&+Q_\text{C}(u|\min(\bar{\theta}_u,\theta^*_{z,\rho}),\min(\pi-\bar{\theta}_u,\theta^*_{z,\rho}),N_\text{C})\\&+Q_\text{J}(u|\min(\pi-\bar{\theta}_u,\theta^*_{z,\rho}),\theta^*_{z,\rho},\lfloor\tfrac{N_\text{J}}{2}\rfloor),\end{aligned} \quad (35)$$

$$\begin{aligned}Q_{\text{U:2D}}(u;z,\rho,N_\text{J},N_\text{C})\triangleq{}&Q_\text{C}(u|0,\min(\bar{\theta}_u,\theta^*_{z,\rho}),\lceil\tfrac{N_\text{C}}{2}\rceil+i_1N_\text{J}+i_2\lfloor\tfrac{N_\text{C}}{2}\rfloor)\\&+Q_\text{J}(u|\min(\bar{\theta}_u,\theta^*_{z,\rho}),\min(\pi-\bar{\theta}_u,\theta^*_{z,\rho}),N_\text{J})\\&+Q_\text{C}(u|\min(\pi-\bar{\theta}_u,\theta^*_{z,\rho}),\theta^*_{z,\rho},\lfloor\tfrac{N_\text{C}}{2}\rfloor),\end{aligned} \quad (36)$$

where $\lceil n\rceil$ and $\lfloor n\rfloor$ denote the smallest integer no smaller than n (ceil) and the largest integer no larger than n (floor), respectively, and $i_1=\{1 \text{ if } \theta^*_{z,\rho}<\bar{\theta}_u \text{ or } 0 \text{ otherwise}\}$ and $i_2=\{1 \text{ if } \theta^*_{z,\rho}>\pi-\bar{\theta}_u \text{ or } 0 \text{ otherwise}\}$.

Proof: Consider the relation [10, Eq. (10)]

$$Q(x,y;\rho)=\frac{1}{2}\cdot Q(x|0,\theta^*_{x/y;\rho})+\frac{1}{2}\cdot Q(y|0,\theta^*_{y/x;\rho}), \quad (37)$$

where [19, Eq. (18)]

$$\theta^*_{z,\rho}\triangleq\operatorname{atan}\left(\frac{z\sqrt{1-\rho^2}}{1-\rho z}\right)\in[0,\pi). \quad (38)$$

As previously, we discuss only the first of the above inequalities. From equation (5) and due to the symmetry of sine around $\pi/2$, it can be seen that the second derivative of $\varphi(\theta,x)$ with respect to θ changes sign twice within $[0,\pi]$, namely at $\bar{\theta}_x$ and $\pi-\bar{\theta}_x$, for all $x\geqslant 0$. There are therefore three cases to distinguish.

The first occurs when $\theta^*_{z,\rho}\leqslant\bar{\theta}_u$, in which case $\varphi(\theta,x)$ reduces to a single convex segment over the interval $\langle 0,\theta^*_{z,\rho}\rangle$. The second case is when $\theta^*_{z,\rho}\in\langle\bar{\theta}_u,\pi-\bar{\theta}_u\rangle$, when $\varphi(\theta,x)$ is convex for $\theta\in\langle 0,\bar{\theta}_u\rangle$ and concave for $\theta\in\langle\bar{\theta}_u,\theta^*_{z,\rho}\rangle$. Finally, if $\theta^*_{z,\rho}\in\langle\pi-\bar{\theta}_u\rangle\pi$, $\varphi(\theta,x)$ has two convex segments, over $\langle 0,\bar{\theta}_u\rangle$ and $\langle\pi-\bar{\theta}_u,\theta^*_{z,\rho}\rangle$, and a concave segment within $\langle\bar{\theta}_u,\pi-\bar{\theta}_u\rangle$. These conditions lead[3] to equation (35), and inequality (33) follows immediately from Theorem 1 and equation (37). ■

Corollary 1.3 - Arbitrarily Tight JC-bounds on $Q_M(x,y)$: For $M=n+\frac{1}{2}$, $n\in\mathbb{N}^+$, and $y\geqslant x>0$, the Generalized Marcum Q-function $Q_M(x,y)$ admits the arbitrarily tight bounds[4]

$$Q_M(x,y)\geqslant Q_{\text{JC}}(y-x,N_\text{J},N_\text{C})+Q_{\text{JC}}(x+y,N_\text{J},N_\text{C}), \quad (39)$$

$$Q_M(x,y)\leqslant Q_{\text{CJ}}(y-x,N_\text{C},N_\text{J})+Q_{\text{CJ}}(x+y,N_\text{C},N_\text{J}). \quad (40)$$

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[2] For simplicity we consider here only the first quadrant of the $x-y$ plane. The extension of Corollary 1.2 to other quadrants is straightforward using the recently-discovered extensions of [10, Eq. (10)] found in [11] and [12].

[3] Is is possible to reduce the conditions on the right hand side of inequalities (35) and (36) to bounds on u – as done for inequalities (30) and (31) – by solving the related quartic equation reduced from $\bar{\theta}_u=\theta^*_{z,\rho}$, with given z and ρ. This, however, leads to expressions that are more computationally demanding than testing $\bar{\theta}_u$ against $\theta^*_{z,\rho}$, as suggested.

[4] The extension to $x,y\in\mathbb{R}$ is trivial and will be done in the journal version of the article.

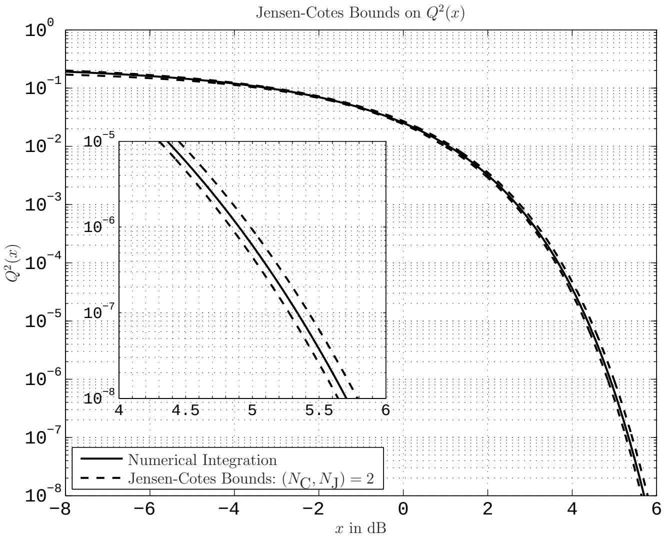

Fig. 5. Jensen-Cotes lower and upper bounds on $Q^2(x)$ given by inequalities (30) and (31), $N_\text{C}=N_\text{J}=2$.

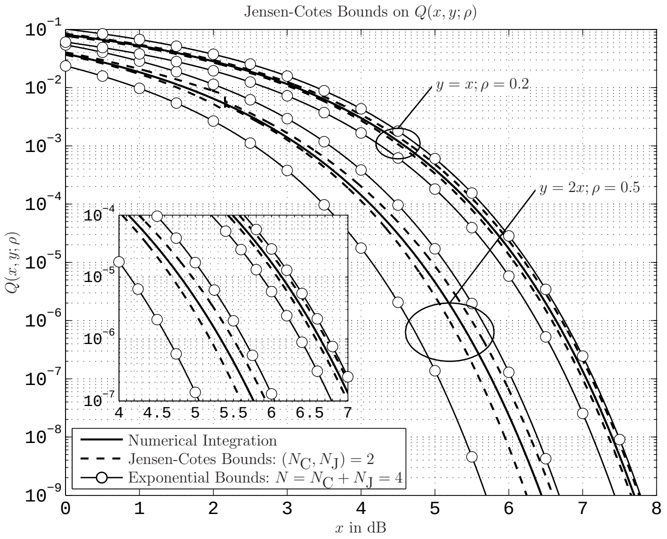

Fig. 6. Examples of Jensen-Cotes lower and upper bounds on $Q(x,y;\rho)$ obtained from inequalities (33) and (34), $N_\text{C}=N_\text{J}=3$.

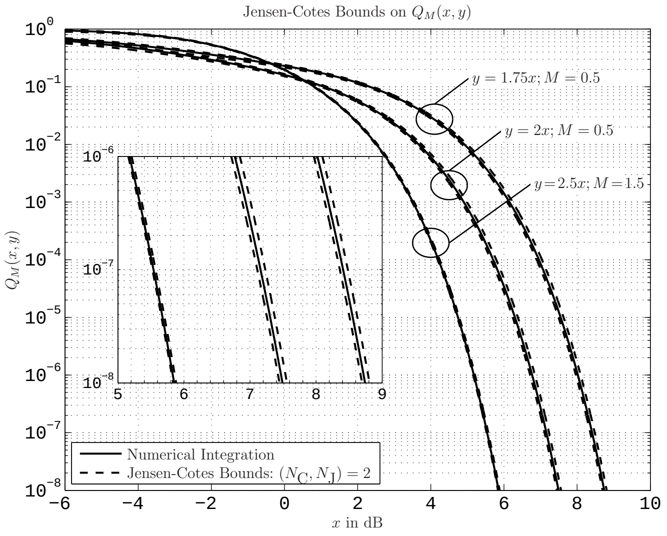

Fig. 7. Examples of Jensen-Cotes lower and upper bounds on $Q_M(x,y)$ obtained from inequalities (39) and (40), $N_\text{C}=N_\text{J}=2$.

Proof: It has been recently shown[5] that for $M = n + \frac{1}{2}, n \in \mathbb{N}^+$, $Q_M(x, y)$ can be re-written as [14, Eq. (11)]

$$Q_M(x,y) = Q(y-x)+Q(x+y)+\Delta Q_M(x,y),\ x \geqslant 0,\ y \geqslant 0, \quad (41)$$

where[6]

$$\Delta Q_M(x,y) \triangleq \frac{1}{x\sqrt{2\pi}} \sum_{k=0}^{M-1.5} \frac{y^{2k}}{2^k} \sum_{q=0}^{k} \frac{(-1)^q (2q)!}{(k-q)!q!} \times \quad (42)$$

$$\times\left[\sum_{i=0}^{2q} \frac{1}{(xy)^{2q-i} i!} \cdot \left[(-1)^i \exp\left(\tfrac{-(y-x)^2}{2}\right) - \exp\left(\tfrac{-(x+y)^2}{2}\right)\right]\right].$$

Inequalities (39) and (40) follow straightforwardly from Theorem 1 and equation (41). ■

Plots of the bounds provided in Corollaries 1.1, 1.2 and 1.3 are shown in figures 5, 6 and 7. The figures confirm that the bounds are very tight even with a few terms only.

As a final remark, we point out that Corollaries 1.2 and 1.3 are particularly attractive, as they offer substantial relief from the computational complexity required to evaluate the 2D joint Gaussian Q-function and the Generalized Marcum Q-function, respectively.

V. CONCLUSION

A new family of arbitrarily tight lower and upper bounds on the Gaussian Q-function $Q(x)$ and a few related functions were presented. The derivation of the bounds exploit a convex-concavity property of the integrand $\varphi(\theta; x)$ of Craig's representation of $Q(x)$, which was proved here.

In addition to arbitrary tightness and computational efficiency, it is found that the proposed Jensen-Cotes bounding method yields lower bounds of equivalent quality to the upper-bounding counter-parts, which to the best of the author's knowledge is unprecedented. Another significant by-product of the method is that various functions related to the Q-function can also be upper- and lower-bounded likewise, examples of which were given.

The proposed bounds are not that tractable and therefore cannot be recommended for further mathematical manipulations. Still, the bounds can be easily implemented and evaluated very fast with standard computers and therefore are excellent for other applications such as those involving optimization of cost-functions containing Q-functions.

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[5] It has further been shown in [14] and [17] that equation (41) can also be used as tight bounds on $Q_M(x, y)$ with $M \in \mathbb{N}^+$. Consequently, an additional implication of Corollary 1.3 is that the JC-inequalities (39) and (40) are also bounds on the generalized Marcum Q-function with integer M, albeit not arbitrarily tightly in these cases.

[6] Alternative, simpler forms of $\Delta Q_M(x, y)$ can be found, $e.g.$, in [17], but this is irrelevant to the result here sought.