Capacity Limits of MIMO Channels

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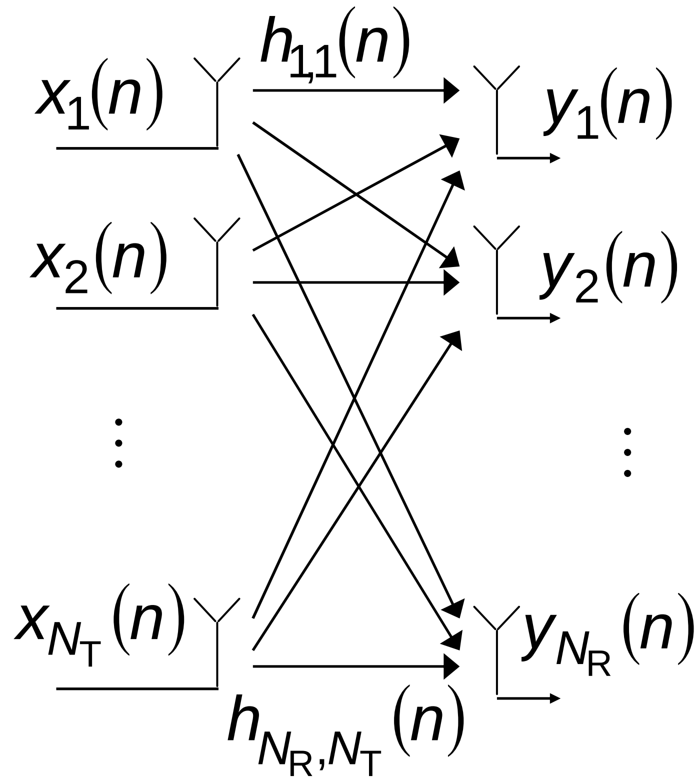
Contents

1. Introduction

- The use of multiple antennas can provide gain due to
  - antenna gain
    - more receive antennas ⇨ more power is collected
  - interference gain
    - interference nulling by beamforming (array gain)
    - interference averaging (to zero) due to independent observations
  - diversity gain against fading
    - receive diversity
    - transmit diversity.
- Information theoretic model of multi-input–multi-output (MIMO) channel is considered.

MIMO Channel Model

- Assume N_T transmit and N_R receive antennae
  - called $N_T \times N_R$ MIMO system.
- Fading radio channels modeled as frequency-flat:
  - fixed
  - time-varying
  - known both/either in the transmitter and/or receiver
    - perfect *channel state information* (CSI)
  - *a priori* unknown.



MIMO channel model.

2. Review of Information Theory

- Information theory (IT) has its origins in analyzing the limits communications.
- Information theory answers two fundamental questions in communication theory:
  - What is the ultimate data compression rate?
    - Answer: entropy.
  - What is the ultimate data transmission rate?
    - Answer: channel capacity.

Basic Concepts

- Assume a discrete valued *random variable* (RV) X with probability mass function $p(x)$.
- The *average information* or *entropy* of RV X:

$$H(X) = -\sum_{x} p(x)\log[p(x)] = -\mathrm{E}[\log p(X)] = \mathrm{E}\left[\log\frac{1}{p(X)}\right].$$

- *Joint entropy* of RV's X and Y:

$$H(X,Y) = -\sum_{x}\sum_{y} p(x,y)\log[p(x,y)] = -\mathrm{E}\{[\log[p(X,Y)]]\}.$$

- *Conditional entropy* of RV Y given $X = x$:

$$H(Y|X) = \sum_{x} p(x)H(Y|X=x) = -\sum_{x}\sum_{y} p(x,y)\log[p(y|x)] = -\mathrm{E}\{\log[p(X,Y)]\}.$$

⇨ Chain rule:

$$H(X,Y) = H(X) + H(Y|X).$$

Mutual Information

- *Mutual information* is the *relative entropy* between the joint distribution and product distribution:

$$I(X;Y) = \sum_x \sum_y p(x,y) \log\left[\frac{p(x,y)}{p(x)p(y)}\right] = \mathrm{E}\left\{\log\left[\frac{p(X,Y)}{p(X)p(Y)}\right]\right\}.$$

$$I(X;Y) = H(X) - H(X|Y) = H(Y) - H(Y|X)$$

$$= H(X) + H(Y) - H(X,Y) = I(Y;X).$$

- Measure of the information one random variable (say, X) contains on the other (Y):
  - If X and Y are independent: $I(X;Y) = 0$ (also “only if”).
  - If $Y = X$: $I(X;X) = H(X)$.
- Differential entropy for continuous RV’s.

Gaussian RV's

- For multivariate, real-valued Gaussian RV's X_1, X_2,..., X_n with mean vector μ and covariance matrix K, the differential entropy is

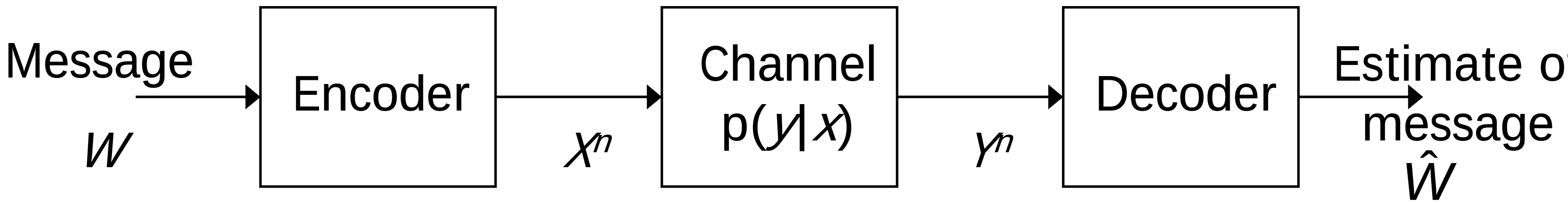
$$h(X_1, X_2, \ldots, X_n) = \frac{1}{2}\log\left[(2\pi e)^n\right]\det(\mathbf{K}).$$

- Gaussian distribution maximizes the entropy over all distributions with the same covariance:

$$h(X_1, X_2, \ldots, X_n) \leq \frac{1}{2}\log\left[(2\pi e)^n\right]\det(\mathbf{K})$$

for any RV's X_1, X_2,..., X_n with equality if and only if they are Gaussian.

Channel Capacity



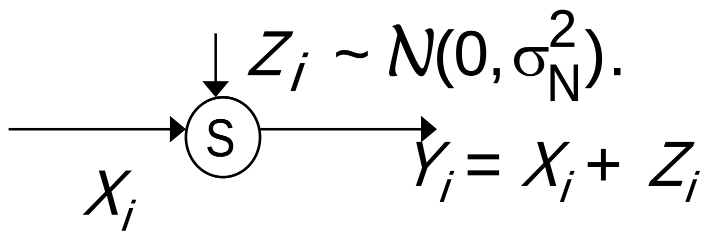
Information theoretic model of a communication system.

- *Channel capacity*: $C = \max_{p(x)} I(X;Y).$

- Code rate R is *achievable*, if there exists a sequence of $(2^{nR}, n)$ codes so that

$$P_{e,\max} \to 0, \text{ as } n \to \infty.$$

Gaussian Channel



The Gaussian channel.

- Channel capacity:

$$C = \max_{\substack{p(x) \\ E(X^2) \le \sigma_S^2}} I(X;Y) = \boxed{\frac{1}{2}\log(1+\gamma)}, \quad \gamma = \frac{\sigma_S^2}{\sigma_N^2}.$$

- Capacity per time unit ((2 W) samples per second):

$$C = \boxed{W \log\left(1 + \frac{P}{N_0 W}\right)}.$$

Parallel Gaussian Channels

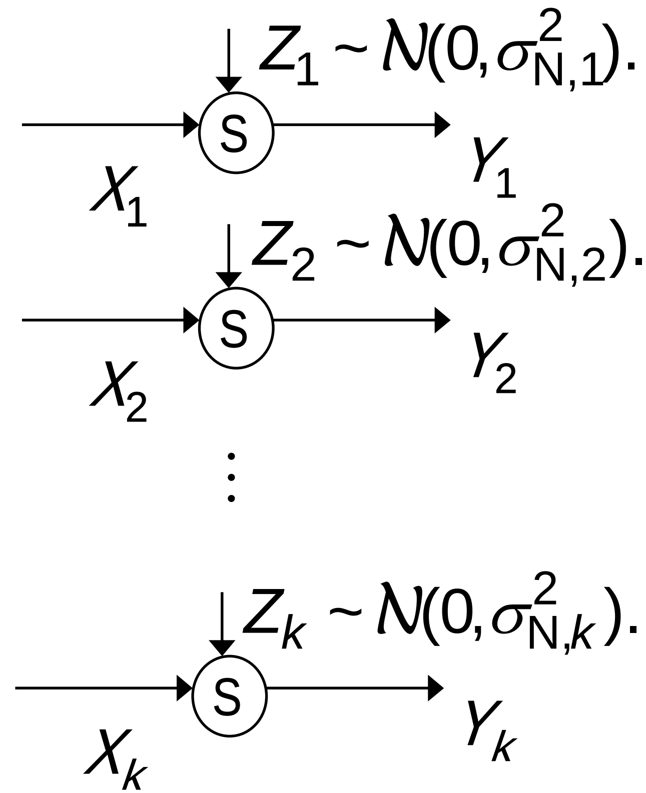
- Capacity:

$$C=\sum_{i=1}^{k}\frac{1}{2}\log\left(1+\frac{\sigma_{S,i}^{2}}{\sigma_{N,i}^{2}}\right)=\sum_{i=1}^{k}\frac{1}{2}\log(1+\gamma_i).$$

- Optimal transmission:

$$\mathbf{X}\sim\mathcal{N}\left(\mathbf{0},\mathrm{diag}\left[\sigma_{S,1}^{2},\sigma_{S,2}^{2},\ldots,\sigma_{S,k}^{2}\right]\right)$$

⇨ *water-filling*.



Parallel Gaussian channels.

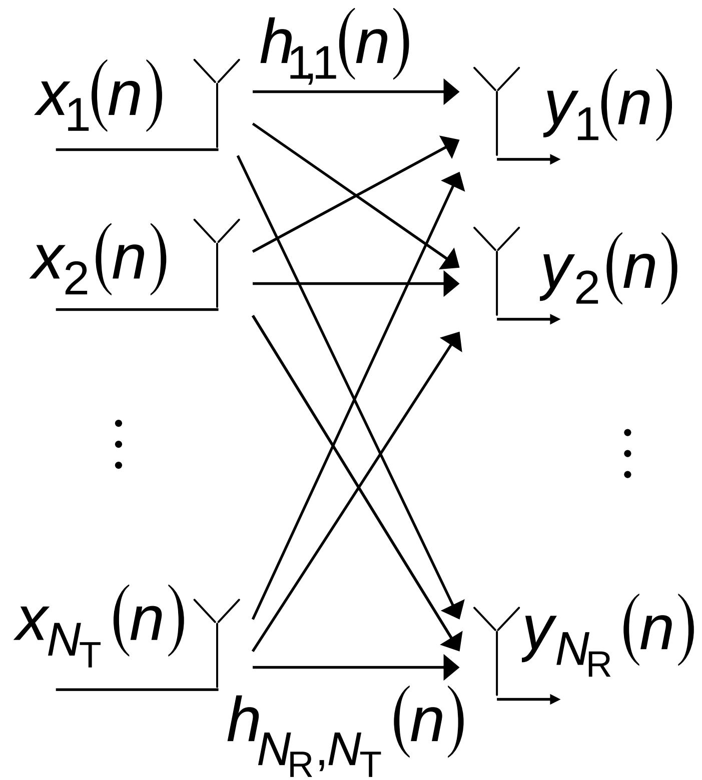
3. Fixed MIMO Channels

- Signal $x_i(n)$ is transmitted at time interval n from antenna i (i=1,2,...,N_T).
- Signal $y_j(n)$ is received at time interval n at antenna j (j=1,2,...,N_R):

$$y_j(n) = \sum_{i=1}^{N_T} h_{ij}(n) x_i(n) + \eta_j(n),$$

where $h_{ij}(n)$ is the complex channel gain with

$$\mathrm{E}\left(\left|h_{ij}(n)\right|^2\right) = 1$$



MIMO channel model.

Matrix Formulation of MIMO Channel Model

- The signal received at all antennas:

$$\mathbf{y}(n) = \mathbf{H}(n)\mathbf{x}(n) + \boldsymbol{\eta}(n),$$

where $\mathbf{x}(n) = \begin{bmatrix} x_1(n) & x_2(n) & \cdots & x_{N_T}(n) \end{bmatrix}^T \in C^{N_T}$,

$$\mathbf{y}(n) = \begin{bmatrix} y_1(n) & y_2(n) & \cdots & y_{N_R}(n) \end{bmatrix}^T \in C^{N_R},$$

$$\mathbf{H}(n) = \begin{bmatrix} h_{1,1}(n) & h_{1,2}(n) & \cdots & h_{1,N_T}(n) \\ h_{2,1}(n) & h_{2,2}(n) & \cdots & h_{2,N_T}(n) \\ \vdots & \vdots & \ddots & \vdots \\ h_{N_R,1}(n) & h_{N_R,2}(n) & \cdots & h_{N_R,N_T}(n) \end{bmatrix} \in C^{N_R \times N_T}.$$

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Noise Model and Power Constraint

- The noise vector

$$\boldsymbol{\eta}(n) = \left[\eta_1(n) \quad \eta_2(n) \quad \cdots \quad \eta_{N_\mathrm{R}}(n)\right]^\mathrm{T} \in C^{N_\mathrm{R}},$$

  satisfies

$$\boldsymbol{\eta}(n) \sim C\mathcal{N}\left(\mathbf{0}, \sigma_\mathrm{N}^2 \mathbf{I}\right).$$

- The transmitted signal satisfies the average power constraint:

$$\mathrm{E}\left(\mathbf{x}^\mathrm{H}(n)\mathbf{x}(n)\right) = \sum_{i=1}^{N_\mathrm{T}} \mathrm{E}\left(\left|x_i(n)\right|^2\right) = \sum_{i=1}^{N_\mathrm{T}} \sigma_{\mathrm{S},i}^2 \le \sigma_\mathrm{S}^2.$$

Singular Value Decomposition

- The MIMO model is a special case of parallel Gaussian channels.
- The channel transfer matrix has *singular value decomposition* (SVD):

$$\mathbf{H} = \mathbf{U}\mathbf{\Lambda}^{1/2}\mathbf{V}^{\mathrm{H}},$$

where

$$\mathbf{U} \in \mathbb{C}^{N_{\mathrm{R}} \times N_{\mathrm{R}}}, \mathbf{V} \in \mathbb{C}^{N_{\mathrm{T}} \times N_{\mathrm{T}}}$$

are unitary matrices, and

$$\mathbf{\Lambda}^{1/2} \in \mathbb{R}^{N_{\mathrm{R}} \times N_{\mathrm{T}}}$$

is a "diagonal" matrix of the singular values of $\boldsymbol{H}$.

Equivalent Channel Model

- Let $\tilde{\mathbf{x}}(n) = \mathbf{V}^{H}\mathbf{x}(n), \quad \tilde{\mathbf{y}}(n) = \mathbf{U}^{H}\mathbf{y}(n), \quad \tilde{\boldsymbol{\eta}}(n) = \mathbf{U}^{H}\boldsymbol{\eta}(n).$
- Since ***U*** and ***V*** are unitary:

$$\mathrm{E}\left(\tilde{\mathbf{x}}^{H}(n)\tilde{\mathbf{x}}(n)\right) \leq \sigma_{S}^{2},$$

$$\tilde{\boldsymbol{\eta}}(n) \sim CN\left(\mathbf{0}, \sigma_{N}^{2}\mathbf{I}\right).$$

⇨ Equivalent channel model

$$\tilde{\mathbf{y}}(n) = \mathbf{\Lambda}^{1/2}\tilde{\mathbf{x}}(n) + \tilde{\boldsymbol{\eta}}(n).$$

"diagonal" matrix of sixe $N_R \times N_T$

⇨ Independent parallel Gaussian channels.

⇨ Capacity achieved with Gaussian input and by water-filling.

Derivation of Channel Capacity

- The rank of matrix $\mathbf{H}$ is $\text{rank}(\mathbf{H}) \le \min(N_R, N_T)$.

⇨ The number of positive singular values is $\text{rank}(\mathbf{H})$.

⇨ The capacity of MIMO AWGN channel:

$$C = \sum_{i=1}^{\text{rank}(\mathbf{H})} \log\left(1 + \frac{\lambda_i \sigma_{S,i}^2}{\sigma_N^2}\right) = \sum_{i=1}^{\text{rank}(\mathbf{H})} \log(1 + \lambda_i \gamma_i), \quad \gamma_i = \frac{\sigma_{S,i}^2}{\sigma_N^2},$$

where the signal powers are solved via water-filling

$$\sigma_{S,i}^2 = \max\left\{0, \left(\frac{\mu - \sigma_N^2}{\lambda_i}\right)\right\}, \; i = 1,2,\ldots,\text{rank}(\mathbf{H}),$$

and μ is chosen so that the power constraint is satisfied or

$$\sum_{i=1}^{\text{rank}(\mathbf{H})} \sigma_{S,i}^2 \le \sigma_S^2.$$

MIMO Channel Capacity for Full–Rank Channel Matrix

- No CSI at the transmitter (and full–rank **H**):

$$C = \log\left[\det\left(\boldsymbol{I}_{N_\mathrm{R}} + \frac{\gamma}{N_\mathrm{T}}\boldsymbol{H}\boldsymbol{H}^\mathrm{H}\right)\right].$$

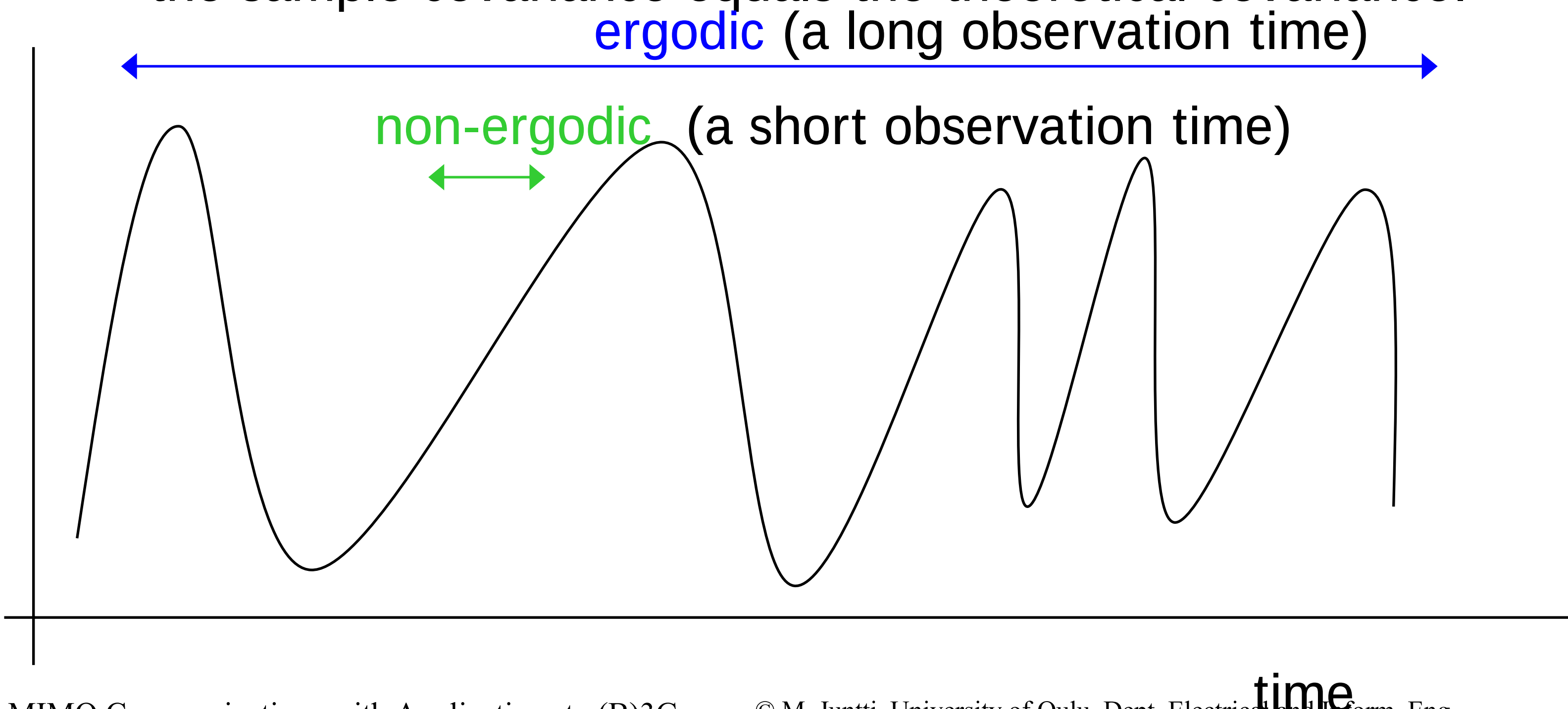
- CSI at the transmitter (and full–rank **H**):

$$C = \max_{\mathbf{Q}} \log\left[\det\left(\mathbf{I}_{N_\mathrm{R}} + \frac{\gamma}{N_\mathrm{T}}\mathbf{H}\mathbf{Q}\mathbf{H}^\mathrm{H}\right)\right],$$

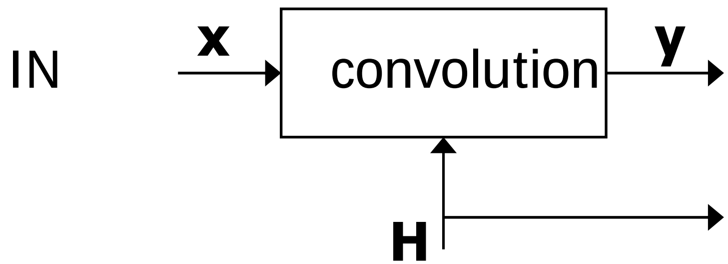
  where **Q** is the covariance matrix of the input vector **x** satisfying the power constraint $\mathrm{tr}(\mathbf{Q}) \leq \sigma_S^2$.
  - No CSI at the transmitter ⇨ **Q** = **I**.

4. Fading MIMO Channels

- The channels are usually assumed to be ergodic: fading is fast enough and gets all realizations so many times that
  - the sample average equals the theoretical mean
  - the sample covariance equals the theoretical covariance.



Fading Channel Model with Perfect Receiver CSI



$$I(\mathbf{x};\mathbf{y},\mathbf{H}) = I(\mathbf{x};\mathbf{H}) + I(\mathbf{x};\mathbf{y}|\mathbf{H}) = I(\mathbf{x};\mathbf{y}|\mathbf{H}).$$

OUT

= 0

RV conditioned on channel realization

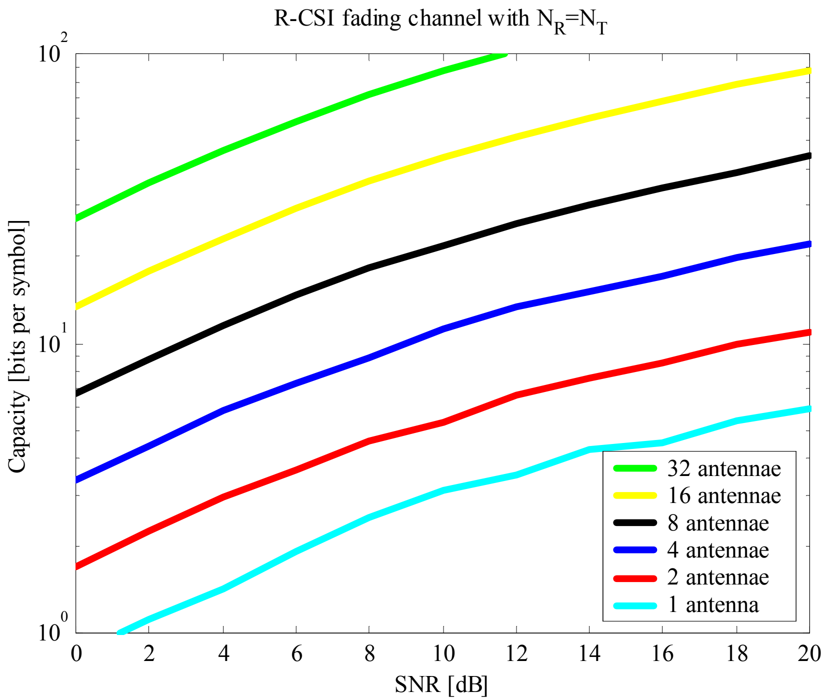
- The effective channel output: the actual channel output **y** and the channel realization **H**.
- Assuming that the channel is memoryless (independent channel state for each transmission), the capacity equals the mean of the mutual information:

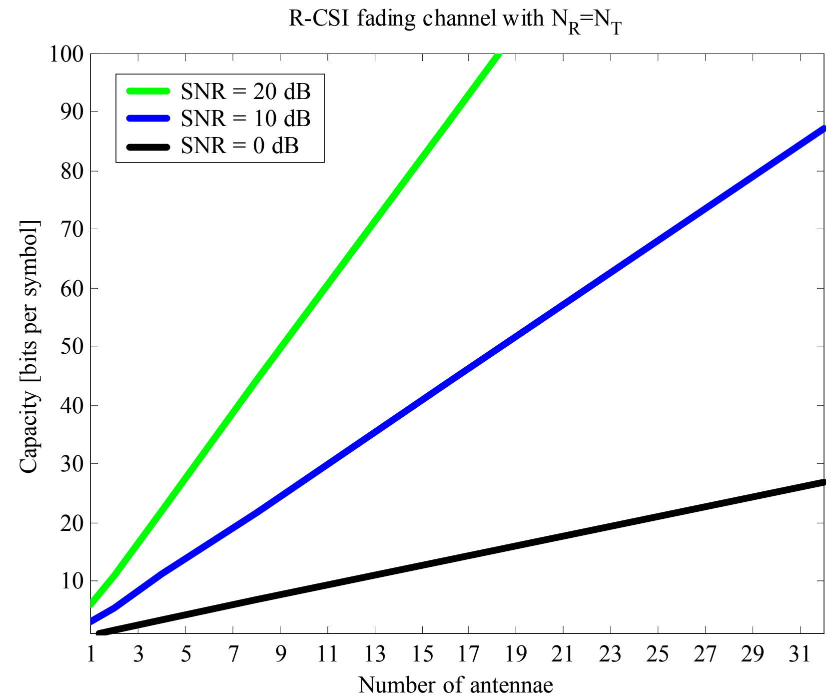
$$C = \mathrm{E}_{\boldsymbol{H}}\left\{\log\left[\det\left(\boldsymbol{I}_{N_\mathrm{R}} + \frac{\gamma}{N_\mathrm{T}}\boldsymbol{H}\boldsymbol{H}^\mathrm{H}\right)\right]\right\}.$$

Capacity Evaluation

- The evaluation of the fading MIMO channel capacity is complicated:
  - Wishart distribution ⇨ Laguerre polynomials [Telatar 1999]
  - bounds [Foschini & Gans 1998]
  - **Monte Carlo computer simulations**
  - random matrix theory ⇨ mutual information tends to Gaussian
    - under development.

Example: $N \times N$ MIMO System



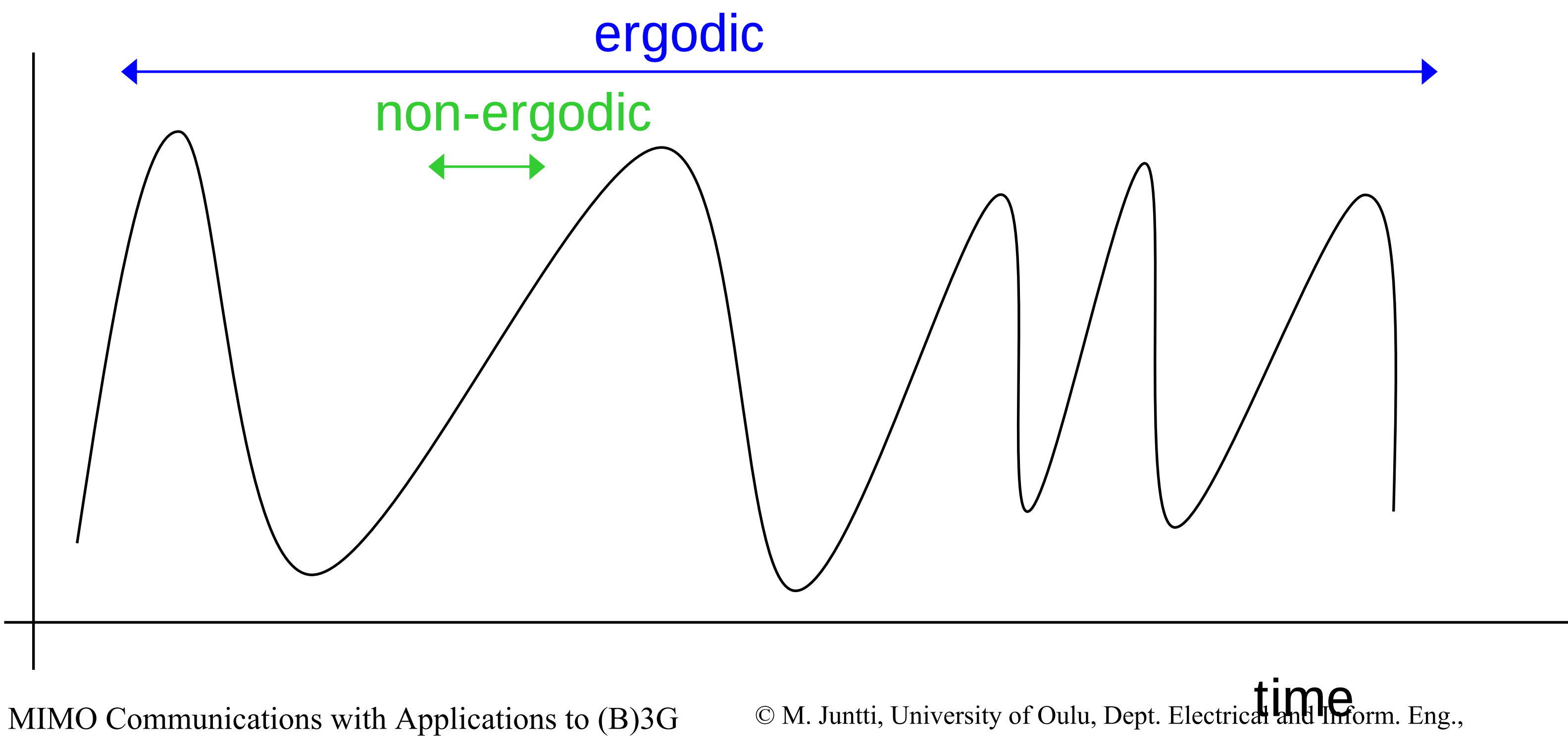


- ⇨ The capacity curves are sifted upwards by introducing more antennae.
- ⇨ The capacity increases linearly vs. the number of antennae.

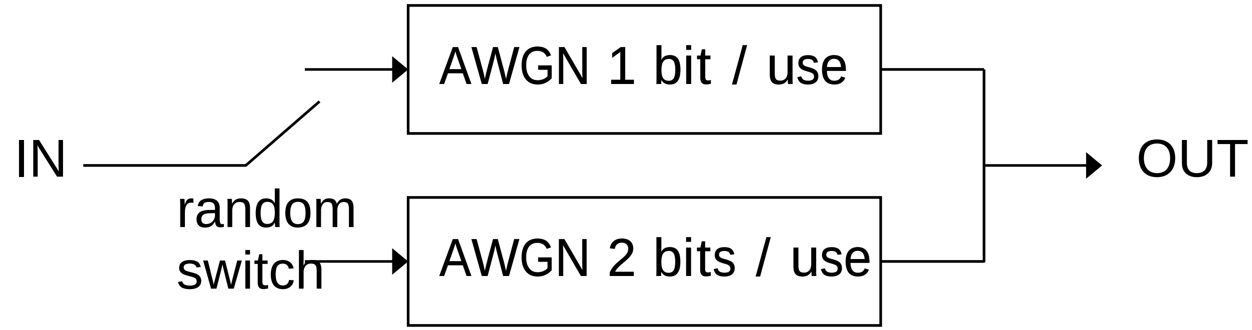
Non-Ergodic Channels

- The channels are not always ergodic: fading can be so slow that it undergoes only some realizations.

⇨ The random process becomes non-ergodic.



Example



- Select one of the channels with equal probability, and keep then fixed.

⇨ Average mutual information is 1.5 bits / channel use.

- However, with probability 0.5 it is not supported.

⇨ The achievable rate $\leq$ 1 bits / channel use.

⇨ **Channel capacity $\neq$ the average maximum mutual information.**

Example: Random and Fixed Channel

- A simple example: generate a channel realization, and keep it fixed during the whole transmission.

⇨ There is a positive probability of an arbitrarily bad channel realization.

⇨ However small a rate, the channel realization may not be able to support it regardless the length of the code word.

⇨ The Shannon capacity of this non-ergodic channel is zero.

⇨ The Shannon capacity is again not equal to the average mutual information.

Outage Probability

- In non-ergodic channels, the capacity is measured by the *probability of outage* for a given rate *R*:

$$P_{\text{out}}(R) = \inf_{\boldsymbol{Q}:\boldsymbol{Q}\geq 0, \text{tr}(\boldsymbol{Q})\leq \sigma_S^2} \Pr\left[I(\boldsymbol{x};\boldsymbol{y}) < R \right]$$

$$= \inf_{\boldsymbol{Q}:\boldsymbol{Q}\geq 0, \text{tr}(\boldsymbol{Q})\leq \sigma_S^2} \Pr\left\{ \log\left[\det\left(\boldsymbol{I}_{N_R} + \frac{\gamma}{N_T} \boldsymbol{HQH}^H \right) \right] < R \right\}.$$

  - Often called capacity versus outage.
- The set-up is encountered in real time applications with transmission delay constraints.
- Similar approach is applicable also for delay constrained communications in ergodic channels.

5. Summary and Conclusions

- AWGN MIMO channels are an extension of parallel Gaussian channels.
  - Another example of parallel channels: channels on different frequencies.
- Introducing both multiple transmit and receive antennae is equivalent to increase in bandwidth.
- The linear capacity increase becomes natural.

$$C = \log\left[\det\left(\boldsymbol{I}_{N_R} + \frac{\gamma}{N_T}\boldsymbol{HQH}^H\right)\right].$$

Fading AWGN MIMO Channel

- Ergodic channels:
  - Channel experiences all its states several times.
  - No delay constraints and/or fast fading.
  - Capacity equals the average mutual information:

$$C = \mathrm{E}_{\boldsymbol{H}}\left\{\log\left[\det\left(\boldsymbol{I}_{N_\mathrm{R}} + \frac{\gamma}{N_\mathrm{T}}\boldsymbol{H}\boldsymbol{H}^\mathrm{H}\right)\right]\right\}.$$

  - Capacity increases linearly with $N_\mathrm{R} = N_\mathrm{T}$.
- Non-ergodic channels:
  - Capacity does not equal the average mutual information.
  - Capacity versus outage probability.

Research Challenges

- Capacity of selective channels
  - time-selective
  - frequency-selective

  with no or imperfect channel state information in the transmitter and the receiver.

⇨ Optimal signal structures (coding and modulation) for real use with issues like
  - amount of training vs. non-coherent detection
  - transceiver complexity constraints
  - limited bandwidth of a non-ideal feedback channel.

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